Inverse Scattering Problem in Hilbert Space

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ABSTRACT
Inverse scattering problems are considered for inhomogeneous dielectric structures with absorption. The retrieval of the complex permittivity is based on the solution of 3D complex-valued integral equations that are reduced to convolution equations with respect to lateral co-ordinates using the plane wave decomposition of Green functions. Then, the inverse problem is reduced to one-dimensional integral equation of the 1-st kind relative to the depth profile of the lateral spectrum of complex permittivity. To solve this ill-posed problem for each pair of spectrum components, a regularization method based on the Tikhonov's theory has been worked out for complex-value functions in Hilbert spaces. Finally, using this method, the desired solution is obtained by the inverse Fourier transform. Numerical results of the algorithm application to the coherent scanning tomography of absorbing targets buried in multilayer media are demonstrated.
Keywords: inverse problem of scattering, scanning tomography, absorbing inhomogeneities

1. INTRODUCTION

The general scheme of the scanning tomography [1] based on the lateral decomposition of 3D integral equations that has been developed for the near-field inverse scattering problem of inhomogeneities in multilayer media [2]. This approach has been used to develop a method of coherent electromagnetic tomography that has been studied numerically, including the case of the perfect lens tomography [3] that can enhance the penetration of the subwavelength sounding. The method of data acquisition at the condition of the fixed emitter-receiver distance, invented in [2], reduces the 3D integral equation for the scattered field to a convolution equation over lateral co-ordinates that makes possible, using the Fourier transform, to reduce the initial 3D problem to the one-dimensional integral equation that should be solved for each pair of spectral components. Starting with the Born approximation, this method can be generalized beyond this approximation. In [2,3], symmetry properties of Green functions have been used to solve the problem for real-value permittivity. This method has been applied in the multi-frequency subsurface tomography of underground objects [4]. The theory has been developed for complex-value permittivity of inhomogeneities, but there were no suitable algorithms to solve the complex integral equation. In this paper, an effective algorithm is developed to solve complex-value Fredholm integral equations of the 1-st kind, and it is used in the tomography simulation of absorbing targets. The possibility to get new information about the material of sounding inhomogeneities is demonstrated.

2. INVERSE PROBLEM OF SCATTERING

Let us consider a multilayer (in z-direction) medium. If the scattering region is embedded in the lth layer of the multilayer structure $\varepsilon = \varepsilon_{0l}$, $\mu = \mu_{0l}$, the complex permittivity in this layer can be written as $\varepsilon(\mathbf{r}) = \varepsilon_{0l} + \varepsilon_1(\mathbf{r})$. The reference (unperturbed) field $\mathbf{E_0}(\mathbf{r})$ that is used in the sounding is determined by the proper Green tensor (that includes near-field components):

$$\mathbf{E}_0(\mathbf{r}) = \int_V \mathbf{j}(\mathbf{r}')\overset{\leftrightarrow}{\mathbf{G}}(x-x', y-y', z, z')d\mathbf{r}', \quad (1)$$

where $\mathbf{j}(\mathbf{r})$ is the current distribution in the source. Using the plane waves' decomposition of (1) over lateral co-ordinates, it is possible to obtain $\mathbf{E_0}(\mathbf{r})$ in any layered medium [2]. In the presence of the scattering region, the electric field can be expressed as the sum of the reference and the scattered fields $\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + \mathbf{E}_1(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E_0}(\mathbf{r}) - \frac{i\omega}{4\pi}\int_V \overset{\leftrightarrow}{\mathbf{G}}(\mathbf{r},\mathbf{r}')\varepsilon_1(\mathbf{r}')\mathbf{E}(\mathbf{r}')d\mathbf{r}', \quad (2)$$

and, in frameworks of perturbation theory, can be determined by known Neumann series starting with the Born approximation:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E_0}(\mathbf{r}) - \frac{i\omega}{4\pi}\int_V \vec{\mathbf{G}}(\mathbf{r},\mathbf{r}')\varepsilon_1(\mathbf{r}')\mathbf{E}_0(\mathbf{r}')d\mathbf{r}' . \tag{3}$$

The solution of the inverse problem requires the solution of the corresponding 3D non-linear integral equation (3) that can be solved iteratively, starting with the Born approximation (2), where a method proposed in [2] can be applied. The main idea of this method is to reduce this equation to a convolution equation over lateral co-ordinates. For this, it is enough to fix the source-receiver spacing $\boldsymbol{\delta}\mathbf{r}$. At this condition, the sounding field structure will be invariable relative to the receiver position that is a very suitable for measurements because all observed variations can be related only to studied inhomogeneities. In this case, it appears possible to reduce the 3D integral equation to 1D integral equation relative to the depth profile of the lateral spectrum of ε_1 [2]:

$$\begin{aligned} E_{1i}(k_x,k_y,z,\omega,\boldsymbol{\delta}\mathbf{r}) &= i\pi\omega\int_{z'}\varepsilon_1(k_x,k_y,z')\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_x+\kappa_x)\delta x+i(k_y+\kappa_y)\delta y}d\kappa_x d\kappa_y \\ &\{\int_{z''}[j_i(-k_x-\kappa_x,-k_y-\kappa_y,z''-z-\delta z)\cdot g_{ij}^{kl}(k_x+\kappa_x,k_y+\kappa_y,z',z'')]g_{ji}^{lk}(\kappa_x,\kappa_y,z,z')dz''\}dz'. \end{aligned} \tag{4}$$

where g_{ji}^{lk} are components the lateral 2D Fourier transform of the Green function $\mathbf{G}$ that determine the contribution of j^{th} component of the source to i^{th} component of the field at the receiver position in the k^{th} layer and the position of the sounded region in the l^{th} layer. The convenient summation over repeated indices ($i, j = x, y, z$) is implied. The proposed sounding at the chosen source-receiver position is sensitive to evanescent components in the scattered field (at $k_\perp^2 = k_x^2 + k_y^2 > |k|^2 = (\omega/c)^2$), hence the subwavelength resolution of this tomography can be achieved. The problem consists of the solution of the one-dimensional Fredholm integral equation of the 1-st kind (4) relative to the vertical profile of permittivity for each pair of lateral spectral components. The kernel of this equation depends on the signal frequency or on the receiver vertical position z of the scanning level. Both parameters can be used in multi-frequency or multilayer methods of tomography [2-4] as depth-of-formation parameters that determine the sensitivity of the sounding in the vertical direction.

To retrieve depth profiles of the complex permittivity perturbation $\varepsilon_1(k_x,k_y,z')$ from the one-dimensional Fredholm integral equation of the 1-st kind (4), a mathematically consistent algorithm based on the Tikhonov's theory of ill-posed problems has been developed. For the case of multi-frequency tomography, the equation (4) is expressed as:

$$E_1(k_x,k_y,\omega) = \int_{z'}\varepsilon_1(k_x,k_y,z')K(k_x,k_y,z',\omega)dz'. \tag{5}$$

It is possible to solve this equation, using the generalized discrepancy principal on complex Hilbert spaces (see, for example, in [5]). In operator form:

$$\mathbf{K}\varepsilon_1 = E_1^\delta , \tag{6}$$

where E_1^δ is data vector measured with errors δE_1 that satisfy

$$\delta E_1^2 = \sup\left\|\mathbf{K}\varepsilon_1 - E_1^\delta\right\|_{L_2}^2 = \sup(\frac{1}{\Delta\omega}\int_V [E_1(\omega) - E_1^\delta(\omega)]^2\, d\omega) , \tag{7}$$

where $E_1(\omega)$ corresponds to the exact solution $\varepsilon_1(z)$, $\Delta\omega$ is the frequency band of analysis. It is necessary to take into account errors of the kernel related to discretization as well as to the error of the Born approximation:

$$\delta_h^2 \le \sup\left\|\mathbf{K}_h\varepsilon_1 - \mathbf{K}\varepsilon_1\right\|_{L_2}^2 = \sup\left\|\mathbf{K_h}\varepsilon_1 - E_1\right\|_{L_2}^2 , \tag{8}$$

where $\mathbf{K}_h$ corresponds to the approximate kernel. Both kinds of error lead to the incompatibility of data with the equation that should be taken into consideration as an additional contribution to errors:

$$\delta_\mu^2 = \inf\left\|\mathbf{K_h}\varepsilon_1 - E_1^\delta\right\|_{L_2}^2 \le \delta E_1^2 + \delta_h^2 \tag{9}$$

The approximate solution ε_1^α is obtained by minimizing of the smoothing functional

$$\mathbf{M}^\alpha(\varepsilon_1) = \left\|\mathbf{K_h}\varepsilon_1^\alpha - E_1^\delta\right\|_{L_2}^2 + \alpha\left\|\varepsilon_1\right\|_{W_2^1}^2 . \tag{10}$$

where the regularization parameter α is obtained from the one-dimensional nonlinear equation of the generalized discrepancy

$$\rho(\alpha) = \left\|\mathbf{K_h}\varepsilon_1^\alpha - E_1^\delta\right\|_{L_2}^2 - \delta^2 = 0 , \tag{11}$$

In above expressions $\|x\|$ is a norm of a function as an element of the complex Hilbert functional space L_2 (the space of square-integrable functions) or W_{21} (the complex Hilbert space of square-integrable complex-value functions with square-integrable generalized derivatives). Particularly, in the expression (10), one has

$$\left\|\varepsilon_1\right\|_{W_2^1}^2 = \frac{1}{\Delta z}\int_{z_{\min}}^{z_{\max}} \left\{ \varepsilon_1^2(z) + \left[\Delta z \frac{d\varepsilon_1}{dz}\right]^2 \right\} dz, \tag{12}$$

$\Delta z = z_{\max} - z_{\min}$. The error parameter $\delta^2 = (\delta E_1 + \delta_h)^2 + \delta_\mu^2$ includes all above noted error components. To minimize the functional (10), the method of conjugate gradients projection has been applied that achieves the minimum at the finite number of steps. The regularization parameter is a monotonous function of δ, so the chord method was applied to solve (11). The main advantage of the developed approach is the convergence of the approximate solution to the exact solution in W_2^1-space (known also as the Sobolev's space) at $\delta \to 0$ in the integral metric L_2. According the Sobolev's including theorem, this leads to the uniform convergence of the solution (in C-metric). As it was shown in [5], it makes possible to use results of single numerical simulations for typical cases, and they will be valid for the estimation of the retrieval accuracy in considered conditions. It is important to stress again that other methods inevitably need for fitting, and, hence, they are case sensitive.

At the solution of (5) in the k-space, the regularization parameter is determined by the integral error of measured lateral spectrum of the scattered field. This spectrum error can be estimated by the known integral error of the measured signal using the Plansherel's theorem. Finally, when the problem is solved for each pair of spectral components, the tomography result (desired 3D structure of permittivity) is obtained by the inverse Fourier transform:

$$\varepsilon_1(x,y,z) = \iint \varepsilon_1(k_x,k_y,z)\exp(ik_x x + ik_y y)dk_x dk_y . \tag{13}$$

3. NUMERICAL SYMULATION

Using the described algorithm to solve the equation (5), we have realized the method of multifrequency tomography based on the 2D lateral scanning of $E_{1x}(x,y,f)$ (where $f=\omega/2\pi$) at five frequencies $f_i = f_1 i,\ i = 1...5$ in the region 1.5 – 7.5 GHz for the three-layered dielectric medium (see in Fig.1). The scheme with the source-receiver position in the waveguiding layer 2 with water-like dielectric parameters is considered. Inhomogeneous absorbing targets are located in layer 3 that has “living tissues” dielectric parameters.

Two different targets are studied: one has a continuous distribution of the complex permittivity with the maximum at $z = z_m$; the second is a homogeneous absorbing dielectric of a parallelepiped form. Different distributions have been chosen to simulate real and imaginary parts of the complex permittivity of the continuous target – the maximum of the ring-shaped distribution of the imaginary part is located outside the main part of the real part distribution, so it is possible to see both distributions in the single image.

All sizes are given at the scale of the shortest wavelength λ_5=0.43 cm in the layer 2 (arrow marked in Fig.1). This wavelength in the layer 2 is much less than the corresponding free-space wavelength, so we achieve a better resolution (the spectrum of evanescent waves is broader). In our simulation, the point source $\mathbf{j}(\mathbf{r}'' - \mathbf{r} - \boldsymbol{\delta}\mathbf{r}) = j_z\delta(\mathbf{r}'' - \mathbf{r} - \delta z\mathbf{z}_0)\mathbf{z}_0$ is scanning over *x*, *y* in the layer 2 in the plane $z = z_s = -0.1\lambda_5$, i.e. below the interface and at the vertical distance $\delta z = -0.1\lambda_5$ from the point, where the *x*-component of the scattered field $E_{1x}(x,y,z_s - \delta z)$ is measured at five frequencies. To simulate measurement errors, the normally distributed random noise (5% of the scattered field in the integral metric) is added.

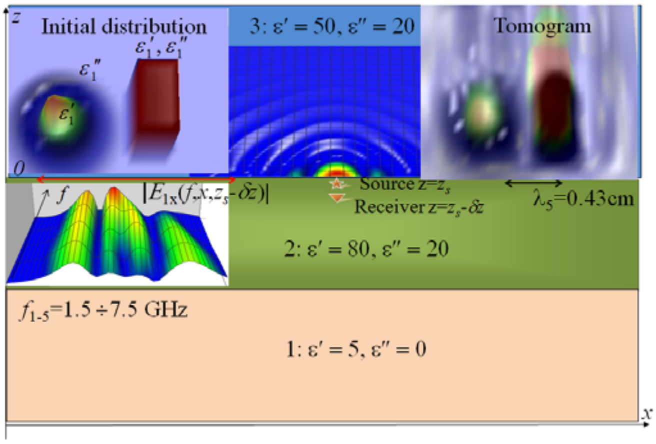

Figure 1. Perfect lens tomography (vertical section). Scheme and simulation results of multifrequency near-field tomography for the three-layered medium (vertical section y = 0. Layer 1: (left) range of frequencies is given. Layer 2: (left) frequency distribution of the "measured" scattered field in the arrow marked range of scanning; (center) source-receiver system. Layer 3: (left) initial distribution of inhomogeneities; (center) z-component of the probing field; (right) retrieved distribution of inhomogeneities (tomogram).The imaginary part of the permittivity can be seen as a grayscale shadow around the real part of the permittivity of the continuous target.

Results of the tomography in the horizontal plane are shown in Fig.2. It is seen that the horizontal resolution of tomography is yet better than that in the vertical direction.

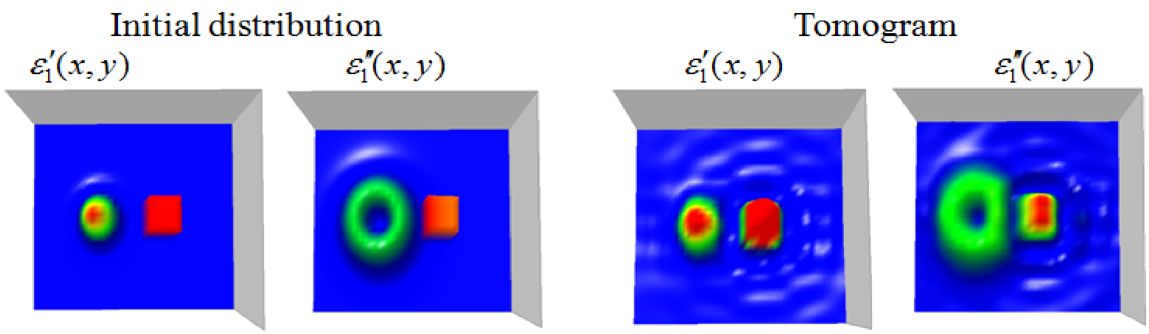

Figure 2. Simulation of multifrequency near-field tomography for the three-layered medium

(horizontal section z = z_m)

4. CONCLUSIONS

The feasibility of the proposed method of the coherent multi-frequency tomography for absorbing targets is shown. The developed method makes possible to obtain additional information about the composition of sounded objects.

ACKNOWLEDGEMENTS

This work was supported by Russian Foundation for Basic Research (grant *No*. 08-02-00117) and by Program of Physical Sciences Department of the Russian Academy of Sciences.

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