Inverse Scattering Problems in Subsurface Diagnostics of Inhomogeneous Media

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ABSTRACT
Methods of electromagnetic sounding, tomography and holography of subsurface dielectric inhomogeneities based on measurements of scattered field are considered. Two main statements of inverse scattering problem are studied: first, based on the solution of the nonlinear integral equation for the scattered field that can be solved iteratively beginning with the Born approximation, and second, based on the dual regularization method – a Lagrange approach in the theory of nonlinear ill-posed problems. Some of these methods have been studied experimentally.

Keywords: electromagnetic sounding, tomography, inverse scattering problem.

1. INTRODUCTION

By now, inverse problems are widely used in remote sensing and non-destructive diagnostics (see, for example, in [1]). Among them, inverse scattering problems can be also applied in various methods of electromagnetic tomography 3D distributions of media parameters [2-5] and in profile retrieval of one-dimensional inhomogeneities [6-10]. In frameworks of electromagnetic perturbation theory, inverse scattering problems in has been reduced to the non-linear integral equation that can be solved iteratively at each step as linear Fredholm integral equations of the 1st kind, beginning with the Born approximation using, for example, Tikhonov's method of generalized discrepancy. However, there are serious limitations of such approach for large perturbations, when the Born approximation is inapplicable as the first guess of iterative method. To overcome these restrictions of perturbation theory, the new method of dual regularization based on the Lagrange approach in the optimization theory has been applied to solve such problems using directly initial Maxwell equations, and results of this application to one-dimensional problems show its ability to retrieve very strong variations of sounded parameters. Here we demonstrate both approaches in the statement and solution of various one- and three-dimensional inverse scattering problems.

2. INVERSE SCATTERING PROBLEMS: THEORY AND SOLUTIONS

Let us consider quite a generous case of a multilayer (in z-direction) medium. If the scattering region is embedded in the l^{th} layer of the multilayer structure $\varepsilon = \varepsilon_{0l}$; the complex permittivity in this layer can be written as $\varepsilon(\mathbf{r}) = \varepsilon_{0l} + \varepsilon_1(\mathbf{r})$. In Maxwell equations

$$\nabla \times \mathbf{E} = \frac{i\omega}{c}\mathbf{H}, \tag{1}$$

$$\nabla \times \mathbf{H} + i\frac{\omega}{c}\varepsilon_i\mathbf{E} = -i\frac{\omega}{c}\varepsilon_1\mathbf{E} + \frac{4\pi}{c}\mathbf{j} = \frac{4\pi}{c}(\mathbf{j}_{eff} + \mathbf{j}), \tag{2}$$

where ω is the frequency, c is velocity of light, it is possible to consider the first term in the right-hand side of (2) as an effective current source $j_{eff} = -i\dfrac{\omega}{4\pi}\varepsilon_1\mathbf{E}$ of the scattered field. Then, representing the total electric field in l^{th} layer as a sum of reference (probing) and scattered fields $\mathbf{E}(\mathbf{r}) = \mathbf{E_0}(\mathbf{r}) + \mathbf{E}_1(\mathbf{r})$, obtain corresponding expressions that solve the direct problem of electrodynamics from the Fredholm integral equation of the 2nd kind:

$$E_i^k(\mathbf{r}) = E_{0i}^k(\mathbf{r}) + E_{1i}^k(\mathbf{r}) = E_{0i}^k(\mathbf{r}) - \frac{i\omega}{4\pi}\int_{V'} \varepsilon_1(\mathbf{r}')E_j^l(\mathbf{r}')G_{ji}^{lk}(x-x', y-y', z, z')d\mathbf{r}', \tag{3}$$

$$E_{0i}^l(\mathbf{r}) = \int_{V'} j_j^k(\mathbf{r}')G_{ji}^{kl}(x-x', y-y', z, z')d\mathbf{r}', \tag{4}$$

$$E_{1i}^k(\mathbf{r}) = -\frac{i\omega}{4\pi}\int_{V'} \varepsilon_1(\mathbf{r}')E_j^l(\mathbf{r}')G_{ji}^{lk}(x-x', y-y', z, z')d\mathbf{r}'. \tag{5}$$

where $G_{ij}^{kl} = {}^{\|}G_{ij}^{kl} + {}^{\perp}G_{ij}^{kl}$, $G_{ji}^{lk} = {}^{\|}G_{ji}^{kl} + {}^{\perp}G_{ji}^{kl}$ are components of Green tensors that are sums of terms for TE ($\perp$) and TH($\|$) - polarizations; j_i is source current distribution (for brevity, we use mainly the same notations for Fourier-transformations of these parameters). The convenient summation over repeated indices (*i, j = x, y, z*) is implied in (3-5). The solution of (3) can be obtained iteratively, beginning with the Born approximation (the first term of the Neumann series).

The statement of inverse scattering problems can be based on equation (5) considered as a non-linear integral equation of the 1st kind with a 6D kernel, and such problems are yet more complicated. As in the above-noted direct problem, it is the evident way to begin the solution using the Born approximation:

$$E_{1i}^{k}(\mathbf{r}) = -\frac{i\omega}{4\pi}\int_{V'} \varepsilon_1(\mathbf{r}')E_{0j}^{l}(\mathbf{r}')G_{ji}^{lk}(x-x', y-y', z, z')d\mathbf{r}' . \tag{6}$$

However, the solution of this 3D equation leads to strong limitations of the grid size used at calculations and, hence, to limitations of the achievable resolution. In some methods of subsurface tomography (radiometry, impedance, low-frequency sounding of earth crust [2], total-internal-reflection tomography [3]) such problems have been reduced one-dimensional integral equations by 2D Fourier transform over transversal co-ordinates. This approach has been developed in [4] for the scheme of measurements with the fixed source-receiver vector $\boldsymbol{\delta}\mathbf{r}$, when the structure of the probing field is invariable relative to the receiver position, and it appeared possible to express the *k*-space spectrum (2D inverse Fourier transform over *x* and *y*) of the scattered field in k^{th} layer in frameworks of the Born approximation:

$$\begin{aligned} E_{1i}(k_x, k_y, \omega, z, \boldsymbol{\delta}\mathbf{r}) = -4\pi^3 i\omega \int_{z'} \varepsilon_1(k_x, k_y, z')\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-i\kappa_x\delta x - i\kappa_y\delta y} \\ \times\int_{z''}[j_i(\kappa_x, \kappa_y, z''-z-\delta z, \omega)G_{ij}^{kl}(\kappa_x, \kappa_y, z'', z', \omega)]G_{ji}^{lk}(\kappa_x+k_x, \kappa_y+k_y, z', z, \omega)d\kappa_x d\kappa_y dz''\}dz', \end{aligned} \tag{7}$$

Based on the solution of (7), algorithms of microwave multifrequency tomography and perfect lens multilevel scanning tomography have been worked out and studied in numerical simulation [4]. To apply the multifrequency algorithm, it should be taken into account that variations of complex amplitudes of the received signal *s* are expressed by the convolution of the instrument function **F** of the receiver and the scattered field $\mathbf{E}_1$:

$$s(\mathbf{r}_r) = \int \mathbf{E}_1(\mathbf{r}')\mathbf{F}(x_r - x', y_r - y', z_r, z')dx'dy'dz' , \tag{8}$$

where $\mathbf{r}_r$ is the vector determining the receiver position. It leads to a convolution equation:

$$s(\mathbf{r}_r, \omega) = \int \varepsilon_1(\mathbf{r}')K(x_r - x', y_r - y', z_r, \omega, z')dx'dy'dz' . \tag{9}$$

So, the transversal spectrum of measured signal variations measured at fixed z_r is obtained as:

$$s(k_x, k_y, \omega) = \int_{z'} \varepsilon_1(k_x, k_y, z')K(k_x, k_y, z', \omega)dz, \tag{10}$$

This equation has been used in our algorithm of multifrequency microwave subsurface tomography that has been developed and studied in numerical simulation of inhomogeneities in multilayer living tissues [4]; however, in our first attempts of its experimental application, it appeared difficult to recognize sounded subsurface objects on the measured image of $s(x, y, \omega)$ against the noise produced by the surface scattering. Nevertheless, we have found that it is possible to obtain much better images of subsurface targets, using the transformation of multifrequency data to the synthesized pseudopulse [5]:

$$s(x_r, y_r, t) = \int_0^{\infty} s(x_r, y_r, \omega)\exp(i\omega t)d\omega = \int \varepsilon_1(\mathbf{r}')K(x_r - x', y_r - y', z_r, t, z')dx'dy'dz' \tag{11}$$

that can be represented in dependence on the effective depth parameter z_s according $s(x_r, y_r, z_s) = s(x_r, y_r, t = -2z_s \operatorname{Re}\sqrt{\varepsilon_0} / c)$ (taking into account the light velocity in a medium and signal path to and from a scattering element):

$$s(x_r, y_r, z_s) = \int \varepsilon_1(\mathbf{r}')K(x_r - x', y_r - y', z_r, z_s, z')dx'dy'dz', \tag{12}$$

The clear visualization of targets is obtained using 2D images of $|s(x_r, y_r, z_s)|$. The strong maximum observed at every point of the scanning region (x_r, y_r) marks the value of z_s that corresponds to surface scattering (that is responsible for the bad quality of data at separate frequencies). This success suggests us to make similar transformations in equation (10):

$$s(k_x,k_y,t)=\int_0^\infty s(k_x,k_y,\omega)\exp(i\omega t)d\omega\,, \tag{13}$$

where $s(k_x,k_y,z_s)=s(k_x,k_y,t=-2z_s\,\mathrm{Re}\sqrt{\varepsilon_0}/c)$, where the integration is, of course, over available frequency band $\Delta\omega$. It leads to the new equation that relates the complex permittivity spectrum to the complex-valued synthesized pseudopulse of the signal lateral spectrum:

$$s(k_x,k_y,z_s)=\int_{z'}\varepsilon_1(k_x,k_y,z')K_1(k_x,k_y,z',z_s)dz'\,, \tag{14}$$

$$K_1(k_x,k_y,z',z_s)=K_1(k_x,k_y,z',t=-2z_s\,\mathrm{Re}\sqrt{\varepsilon_0}/c)\,,$$

$$K_1(k_x,k_y,z',t)=\int_0^\infty K(k_x,k_y,z',\omega)\exp(i\omega t)d\omega\,.$$

This transformation leads to the depth dependence of $K_1(k_x,k_y,z',z_s)$ with maxima that can explain the observed depth selectivity and resolution of pseudopulse images and provide better solution results in comparison with the exponential kernel of initial equation (10). It is suitable to mention here that this pseudopulse approach has been also successfully applied in the multifrequency diagnostics of one-dimensional diffuse inhomogeneities in periodic structures [10], and we propose to apply this approach to low-frequency geomagnetic diagnostics.

To solve the Fredholm integral equation (14), the algorithm based on the generalized discrepancy principle in the complex Hilbert space W_2^1 [4] has been applied here to retrieve tomography images of subsurface inhomogeneities with the complex-valued distribution of permittivity. From the solution of (14), the desired 3D structure of permittivity (tomogram) is obtained by the 2D inverse Fourier transform:

$$\varepsilon_1(x,y,z)=\iint\varepsilon_1(k_x,k_y,z)\exp(ik_xx+ik_yy)dk_xdk_y\,. \tag{15}$$

Then, for targets with a homogeneous internal structure, it is possible to obtain their shape (i.e. to solve the problem of computer holography) as two functions $x_1(y,z)$, $x_2(y,z)$ in each section z = *const*, using the *k*-space solution of (13) (its inverse 1D Fourier transform $\varepsilon_1(k_x,y',z)=\int_{-\infty}^{\infty}\varepsilon_1(k_x,k_y,z)\exp(ik_yy')dk_y$) from the complex-value transcendent equation:

$$\varepsilon_1(k_x,y,z)=\frac{\varepsilon_1^0}{2\pi ik_x}\left(e^{-ik_xx_1(y,z)}-e^{-ik_xx_2(y,z)}\right). \tag{16}$$

obtained in [5] that is equivalent to the system of two real-valued equations. It should be mentioned that this equation is overdetermined: it can be solved at each value of k_x. In Fig.1 an example of such holography from [5] is shown for a foam target in the sandy ground.

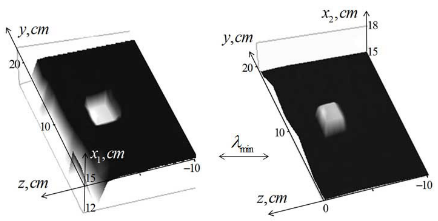

Figure 1. Holography images of a foam target [5].

A good target\s localization achieved in the pseudopulse approach makes it easier to deal with the inverse scattering problem beyond the Born approximation. The possible statement of this problem could be based on the iterative solution of the non-linear integral equation:

$$E_{1i}^k(\mathbf{r})=-\frac{i\omega}{4\pi}\int_{V'}\varepsilon_1^{(n)}(\mathbf{r}')[E_{0j}^l(\mathbf{r}')+E_{1j}^l(\varepsilon_1^{(n-1)},\mathbf{r}')]G_{ji}^{lk}(x-x',y-y',z,z')d\mathbf{r}'\,. \tag{17}$$

However, our results of our study of more simple one-dimensional problems [7-9] demonstrate serious limitations of such iterative approach for large perturbations, when the Born approximation (used as the first guess of iterative method) gives considerable deviations from true solutions. To overcome these restrictions of perturbation theory, the new method of dual regularization based on the Lagrange approach in the optimization theory has been proposed and applied to solve such problems [4-9]. It can be based on initial Maxwell equations, and results of this application to one-dimensional problems of low-frequency sounding of Earth crust conductivity profile and to retrieve profiles of diffuse perturbations of permittivity in multilayer structures of X-ray optics, show its ability to retrieve very strong variations of sounded parameters.

In the dual regularization algorithm for 3D problems proposed here by analogy with those for one-dimensional problems [7-10], it is necessary to satisfy the condition

$$s[\varepsilon_1](k_x,k_y,z_s)=s_0(k_x,k_y,z_s)\,, \tag{18}$$

where $s[\varepsilon_1]$, s_0 are calculated and measured signal, respectively, in the process based on the minimization of the modified Lagrange functional:

$$\begin{aligned} L_\mu(\varepsilon_1,\lambda) \equiv \|\varepsilon_1\|^2 + \int_{z_{s1}}^{z_{s2}} \iint dk_x dk_y \,\mathrm{Re}[\lambda^*(k_x,k_y,z_s)\left(s[\varepsilon_1](k_x,k_y,z_s)-s_0(k_x,k_y,z_s)\right)]dz_s \\ +\mu(\sqrt{\int_{z_{s1}}^{z_{s2}} \iint dk_x dk_y \left(s[\varepsilon_1](k_x,k_y,z_s)-s_0(k_x,k_y,z_s)\right)^2 dz_s} \\ +\int_{z_{s1}}^{z_{s2}} \iint dk_x dk_y \left(s[\varepsilon_1](k_x,k_y,z_s)-s_0(k_x,k_y,z_s)\right)^2 dz_s), \end{aligned} \tag{19}$$

where λ are Lagrangian coefficients, along with the simultaneous maximization of the regularized dual problem

$$\begin{aligned} V_\mu^\alpha = V_\mu(\lambda) - \alpha\|\lambda\|^2 \equiv \min_{\varepsilon_1\in L_2} L_\mu(\varepsilon_1,\lambda) - \alpha\|\lambda\|^2 \to \max, \\ \lambda\in\Lambda_\mu \equiv \{\lambda\in \mathrm{L}_2) : \|\lambda\| \le \mu\}, \end{aligned} \tag{20}$$

where α is the Tikhonov's regularization parameter. The supergradient of this latter functional is expressed explicitly. The saddle point of this process gives the desired solution. At that, it is reasonable to begin the minimization of (19) using as the first guess the solution of (13) obtained in Born approximation. Of course, supercomputers should be used to realize this algorithm. As a next step in the development of inverse scattering methods of subsurface diagnostics, we intend to work out yet more effective and stable dual-regularization algorithms based on sequential approach to Kuhn-Tucker theorem [11].

3. CONCLUSIONS

Statements of inverse scattering problems for applications in various methods of subsurface electromagnetic diagnostics of inhomogeneous media based on the solution of the nonlinear integral equation for the scattered field and on the dual regularization method has been considered.

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