One-Dimensional Inverse Scattering Problem

[1] **P. K. Gaikovich**, [2] **M. I. Sumin**, [1] **K. P. Gaikovich**

[1] *Institute for Physics of Microstructures RAS, GSP-105, Nizhniy Novgorod, Russia*

Tel: +7(831)2581606, Fax: +7(831) 4385555, e-mail: Gaikovich@ipmras.ru

[2]*Nizhniy Novgorod State University, Nizhniy Novgorod, Russia*

e-mail: m.sumin@mm.unn.ru

ABSTRACT

The inverse problem of electromagnetic scattering of media with one-dimensional permittivity distribution is considered. Two approaches are applied in the study. First is based on the solution of non-linear integral equation for the scattered field; second-involves in analysis the Lagrange formalism applied to initial differential equations (Maxwell's equations). Based on the developed theory, solution algorithms have been worked out for diagnostics of subsurface permittivity inhomogeneities into multilayer periodic dielectric structures.

Keywords: inverse scattering problem, permittivity profile, multilayered periodic structures

1. INTRODUCTION

Inverse problems of scattering are widely used in various methods of sounding and tomography of media parameters in electromagnetism, acoustics and quantum mechanics. For one-dimensional (1D) distributions of media parameters, they can be reduced to the known Gelfand-Levitan-Marchenko equation (1). However, such generalizations are inapplicable to layered or absorbing media. This simplest problem for the ultra low-frequency electromagnetic geomagnetic sounding of earth crust permittivity profiles has been formulated firstly by Tikhonov [2] in frameworks of his general theory of ill-posed problems. The frequency dependence of the effective depth of the received signal formation (skin-depth) of measured fields was in use in this method, applied further in the magnetotelluric exploration [3].

In [4-6], in frameworks of electromagnetic perturbation theory, the inverse scattering problem in various statements has been reduced to the non-linear integral equation of the 1[st] kind that should been solved iteratively, beginning with the Born approximation. Based on this equation, the one-dimensional problem of low-frequency earth crust profiling has been solved with the use of Tikhonov's method of generalized discrepancy [7]. Results of the numerical study [7] for low-frequency conductivity sounding have demonstrated serious limitations of such approach for large perturbations, when the Born approximation (first guess of iterative method) is inapplicable. To overcome these restrictions of perturbation theory, the new method of dual regularization based on the Lagrange approach in the optimization theory [8] has been applied in this problem to solve initial Maxwell equations [9]. Results show its ability to retrieve very strong variations of conductivity profiles.

The approach, based on the solution of the non-linear integral equation has been also applied to diagnostics of permittivity inhomogeneities in multilayer periodical structures that are basic elements of the modern X-ray optics [10]. Since their invention in 1976 [11], they have been widely used as reflectors, polarisers and filters in the "soft" X-ray range, where crystals are irrelevant. Their parameters were optimized for different purposes, but some deviations from a desired perfect meander structure appear at the synthesis. For diagnostics of these structures, measurements of X-ray scattering are in use. This method has some obvious advantages: it is noncontact, non-destructive and fast in comparing to an electron microscopy or SIMS (secondary ion mass-spectrometry). One-dimensional structure defects can be described in the terms of permittivity profile by a mirror depth. Here we present results of numerical simulation of the corresponding inverse scattering problem solution based on non-linear integral equation and propose a new method of this problem solution based on the dual-regularization approach.

2. APPROACH BASED ON THE INTEGRAL EQUATION

Following [10], consider a periodic multilayer (in *z*-direction) medium with the period $d = d_1 + d_2$ with a complex permittivity profile $\varepsilon(z) = \varepsilon'(z) + i\varepsilon''(z)$ that is expressed as

$$\varepsilon(z) = \begin{cases} \varepsilon_{01}, \; z < 0 \\ \varepsilon_{02} + \varepsilon_1(z), \; 2id \le z < 2id + d_1 \\ \varepsilon_{03} + \varepsilon_1(z), \; (2i+1)d \le z \le (2i+1)d + d_2 \\ \varepsilon_{04}, \; z > Nd \end{cases}, \; i = 0, 2..., N/2 \qquad (1)$$

where the profile of inhomogeneities $\varepsilon_1(z) = \varepsilon_1(z + d)$ is also periodic. If the distribution of a probing electric field at $\varepsilon_1 = 0$ is $\mathbf{E_0}(\mathbf{r})$, the total field for a structure with inhomogeneities can be expressed as a sum of probing and scattered fields $\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + \mathbf{E}_1(\mathbf{r})$. In the considered case of the reflectometry of one-dimensional media, when plane waves are used as the probing field, the total field is also expressed as $\mathbf{E}(\mathbf{r}) = \mathbf{E}(z)\exp(ik_x x + ik_y y)$, so that:

$$\mathbf{E}(z) = \mathbf{E}_0(z) - \frac{i\omega}{4\pi}\int_{z'} \varepsilon_1(z')\vec{\mathbf{g}}(z',z)\mathbf{E}(z')dz' \tag{2}$$

where the Green tensor $\vec{\mathbf{g}}$ for a multilayer media are obtained using the input impedance formalism [12]. The field distribution can be obtained iteratively from (2) in the form of Neumann series beginning with the Born approximation ($\mathbf{E}(z') = \mathbf{E}_0(z')$ in the second term of the right-hand side in (2)). To solve the corresponding inverse problem, this equation also should be solved beginning with the Born approximation, when it is possible to obtain a quite simple equation for the perturbation of the power reflection coefficient Δr [10]. It depends on the viewing angle and frequency and can be written in a compact form as:

$$\Delta r(\theta,\omega) = \left\langle |R|^2 \right\rangle - \left\langle |R_0|^2 \right\rangle = \left\langle |R_1|^2 \right\rangle + \mathrm{Re}\int_{z'} \varepsilon_1(z') g(\theta,\omega,z')dz' \tag{3}$$

where $g(\theta,\omega,z')$ includes contributions of Green functions for TE and TH polarizations, the total reflection coefficient $|R|^2 = |R_0|^2 + 2\,\mathrm{Re}(R_0^* R_1) + |R_1|^2$ is determined by reflection coefficient of the unperturbed meander structure R_0 (that can be calculated by known parameters of the meander structure d_1, d_2, N and by known dielectric parameters of this structure) and by the coefficient R_1 related to the scattering on inhomogeneities. Because $\varepsilon_1(z)$ is formed due to the mutual penetration of two components of the meander structure, it is reasonable to represent it as $\varepsilon_1(z) = f(z)(\varepsilon_{03} - \varepsilon_{02})$, where the real profile $f(z)$ determines the complex permittivity perturbations. Then, it is possible to use the periodicity property of $\varepsilon_1(z')$ to include the contribution of all the layers in the kernel:

$$\Delta r(\theta,\omega) = \int_d f^{(n)}(z)\,\mathrm{Re}\{2R_0^*(\varepsilon_{03} - \varepsilon_{02})\sum_{i=1}^{\frac{N-1}{2}} g[\theta,\omega,z + (i-1)]\}dz + \left|R_1(f^{n-1}(z))\right|^2 \tag{4}$$

Denoting $\Delta r^n(\theta,\omega) = |R|^2 - |R_0|^2 - \left|R_1(\varepsilon_1^{n-1}(z))\right|^2$, we obtain the desired integral equation of the inverse scattering problem that can be written in a compact form as:

$$\Delta r^{(n)}(\theta,\omega,f^{(n-1)}) = \int_0^d f^{(n)}(z) K(\theta,\omega,z)dz \tag{5}$$

that can be solved iteratively, beginning with $f^{(0)} = 0$ ($|R_1|^2 \ll |R_0|^2$). At each step of the iteration, the equation (5) is a Fredholm integral equation of the 1st kind. To solve this ill-posed problem, we use the Tikhonov's method of generalized discrepancy [4,10].

The proposed method has been applied to the study of the permittivity perturbations in the multilayer X-ray mirrors. It is well-known that depths d_1, d_2 of layers can be determined for these periodic structures from angular reflectometry measurements at a short enough wavelength ($\lambda = 0.154$ nm for the periodic Mo-Si 50-layer structure in [10]), using the best fit. Then, the permittivity inhomogeneities of this structure have been studied using multifrequency reflectometry measurements at longer waves that are more sensitive to the profile variations. The spectral range $\lambda = (12.5 \div 14.5)$ nm has been in use in [10] at the elevation angle $\theta = 85°$, where TE-component is dominated in scattering. As it has been shown in [10], in this spectral range the kernel of (5) has a high depth-frequency variability that ensures the proper sensitivity to profile variation.

In Fig. 1 results of numerical simulation of the profile retrieval from the solution of (5) are given with the same parameters of meander structure as in [10] and at the same level of simulated random errors ($\delta R^2 = 0.015$).

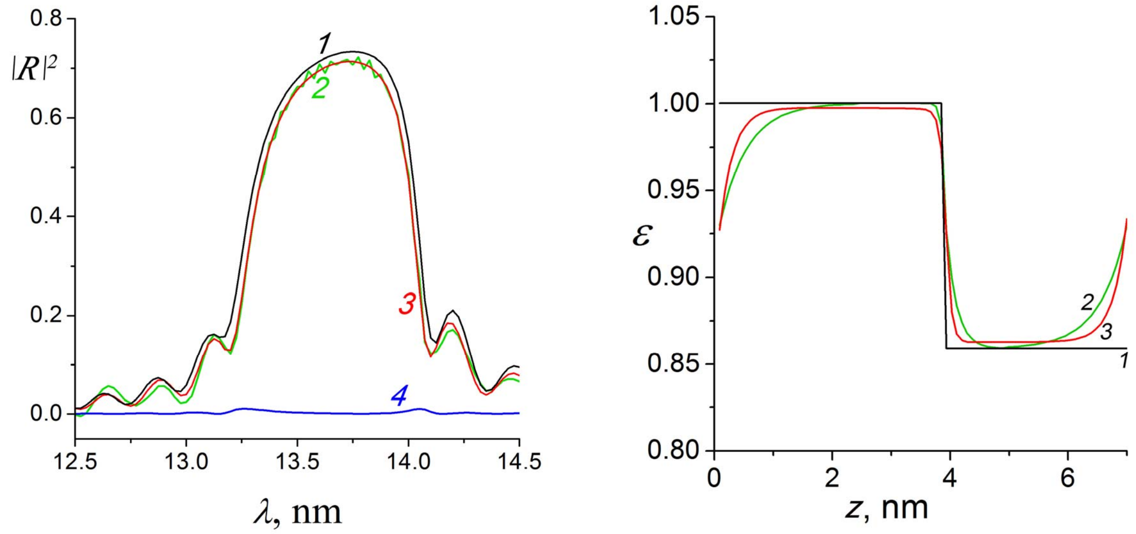

Figure 1: Left,(1) $|R_0(\omega)|^2$ *, (2)* $|R(\omega)|^2 + \delta R^2$*, (3) retrieved* $|R(\omega)|^2$*, (4)* $|R_1(\omega)|^2$ *;*
Right, (1) meander structure, (2) initial profile of $\mathrm{Re}\,\varepsilon(z)$ *,(3) retrieved profile.*

Results demonstrate a good enough quality of the profile retrieval. Calculations show that the correction of the kernel in the next approximation of the perturbation theory has small effect on the solution. However, it was obtained in the simulation that at stronger profile variations, the contribution of the term $|R_1(\varepsilon_1(z))|^2$ (line 4 in Fig. 1), that is neglected at the first step of iterations in (5), can break the algorithm convergence. Because of this reason, we propose here the algorithm of dual regularization method to solve this non-linear ill-posed problem.

3. DUAL REGULARIZATION METHOD

For brevity, consider the scheme of the dual regularization method for measurements of TE-polarization. The differential equation for this component in a dimensionless form is written as

$$\frac{d^2e}{dz^2} + \left(\frac{\omega}{c}\right)^2 [\varepsilon(z) - \cos^2\theta] e = 0 \tag{6}$$

where $E_0 = E(z=0,\, \varepsilon=1),\ \mathrm{Im}\,E_0 = 0,\ \ e = E(z=0,\, \varepsilon(z))/E_0$ .Taking into account that $R(\omega) = |e(z=0)-1|^2$ and changing variables $x_1 = \mathrm{Re}\,e,\ \ x_2 = d\,\mathrm{Re}\,e/dz,\ \ x_3 = \mathrm{Im}\,e,\ \ x_4 = d\,\mathrm{Im}\,e/dz$, $\varepsilon(z) = \mathrm{c} + i\mathrm{d} + (\mathrm{a} + i\mathrm{b}) f(z)$, write the main scheme of the proposed algorithm. To obtain the solution, it is necessary to minimize the functional

$$I_0(f, x_0) \equiv \|f\|^2 + \|x_0\|^2 \to \min \tag{7}$$

at conditions:

$$(f,x) \in \mathrm{D} \equiv \{f : [z_0, z_n] \to \mathrm{R};\ f \in \mathrm{L}_2(z_0, z_n);\ x \in \mathrm{R}^4\},\ I_1(f, x_0)(\omega) \equiv [x_{01} - 1]^2 + x_{03}^2,\ \omega \in [\omega_1, \omega_2],$$
$$I_1(f, x_0) = R(\omega),\ \ (f, x_0) \in \mathrm{L}_2(z_0, z_n) \times R^4 \equiv \mathrm{D},\ f(z+d) = f(z)$$
$$\frac{dx}{dz} = A(f(z))x,\ \ x_i(z_0) = x_{0i} \equiv (x_{01}, x_{01}, x_{01}, x_{01})^*,\ \ \ x(z) \to 0,\ z \to \infty,$$

$$A(f(z)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -(\frac{\omega}{c})^2(\mathrm{c}+\mathrm{a}f - \cos^2\theta) & 0 & (\frac{\omega}{c})^2(\mathrm{d}+\mathrm{b}f) & 0 \\ 0 & 0 & & 1 \\ -(\frac{\omega}{c})^2(\mathrm{d}+\mathrm{b}f) & 0 & -(\frac{\omega}{c})^2(\mathrm{c}+\mathrm{a}f - \cos^2\theta) & 0 \end{pmatrix}$$

In the considered problem it is necessary to use the modified Lagrange function [8]. It can be built as

$$L_\mu(f,x_0,\lambda) \equiv \| f \|^2 + \| x_0 \|^2 + \int_{\omega_1}^{\omega_2} \lambda(\omega)\left[I_1[f,x_0](\omega) - R(\omega)\right] d\omega$$
$$+\mu\left(\sqrt{\int_{\omega_1}^{\omega_2}\left[I_1[f,x_0](\omega) - R(\omega)\right]^2 d\omega} + \int_{\omega_1}^{\omega_2}\left[I_1[f,x_0](\omega) - R(\omega)\right]^2 d\omega\right),\ \lambda \in \mathrm{L}_2(\omega_1,\omega_2),\ \mu > 0 \qquad (8)$$

At a large enough μ, (8) has a guaranteed minimum obtained by gradient minimization. The regularized dual problem that gives the minimum of (7) consists of maximizing the concave functional

$$V_\mu(\lambda) - \alpha \| \lambda \| \equiv \min_{(f,x_0)} L_\mu(f,x_0,\lambda) - \alpha \| \lambda \| \to \max, \quad \lambda \in \Lambda_\mu \equiv \{\lambda \in \mathrm{L}_2(\omega_1,\omega_2) : \| \lambda \| \le \mu\}, \qquad (9)$$

where α is the regularization parameter on the Hilbert space $L_2(\omega_1,\omega_2)$, using the explicit formula for the supergradient $\partial V_\mu(\lambda)/\partial\lambda = conv\{[I_1[f,x_0](\omega) - R(\omega)]\,|_{f=f_\mu[\lambda],x_0=x_{0\mu}[\lambda]}$.

4. CONCLUSIONS

The one-dimensional inverse scattering problem based on the non-linear integral equation has been studied in numerical simulation for permittivity profiling of periodic structures. New method of dual regularization based on the solution of initial Maxwell's equations has been proposed to overcome available difficulties in solution.

ACKNOWLEDGEMENTS

This work was supported by grant of Education and Science Ministry *No*.2.1.1/3927, by grant of Federal Program *No*.NK-13P-13, and by Program of Physical Sciences Department of the Russian Academy of Sciences.

REFERENCES

[1] Gelfand I.M., Levitan B.M.: On the determination of a differential equation from its spectral function, *Amer. Math. Soc. Transl. (2),* **1**, p. 253, 1955.

[2] Tikhonov A.N.: *Solution of Ill-Posed Problems*, Winston, New York, 1977.

[3] Zhdanov M.S., Keller G.: *The Geoelectrical Methods in Geophysical Exploration*, Elsevier, Amsterdam, 1994.

[4] Gaikovich K.P.: *Inverse Problems in Physical Diagnostics*, Nova Science Publishers Inc., New York, 2004.

[5] Gaikovich K.P.: Subsurface near-field scanning tomography, *Physical Review Letters*, **98**, p.183902, 2007.

[6] Gaikovich K., Vorgul I., Marciniak M.: On retrieval of permittivity profile, *in Proc. ICTON* 2001, Cracow, Poland, 18-21, pp.238-240, June 2001.

[7] Gaikovich K.P. Ultra low frequency sounding and tomography of earth crust, in *Proc. 3^{d} Int. Conf.: Ultrawideband and Ultrashort Impulse Signals*, Sevastopol, Ukraine, p. 294, 2006.

[8] Sumin M.I.: Regularized Dual Method for Nonlinear Mathematical Programming, *Comput. Math. Math. Phys.*, **47**, p. 760, 2007.

[9] Gaikovich K.P., Gaikovich P.K., Galkin O.E., Sumin M.I.: Dual regularization in one-dimensional inverse scattering problem, *in Proc. 5th International Conference "Ultrawideband and Ultrashort Impulse Signals"*, Sevastopol, Ukraine, p.90, September 6-10, 2010.

[10] Barisheva M. M., Gaikovich K. P., Gaikovich P. K., Polkovnikov V. N., Vainer Yu. A., Zuev S. Yu.: Reflectometry Sounding of Inhomogeneities in Periodic Multilayer Structures. *in Proc. ICTON* 2010, Munich, Germany, 1, pp.Tu.P5, June 27-July 2010.

[11] E. Spiller.: Reflective multilayer coatings in the far UV region, *Appl. Opt*., **15**, p.2333, 1976.

[12] Gaikovich K.P., Gaikovich P.K.: Inverse problem of near-field scattering in multilayer media, *Inverse Problems*, **26**, *no*.12, p. 125013, 2010.