Spectral approach to correlation theory of thermal regime and thermal emission of the medium with random boundary conditions

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Abstract- The results of applying stochastic spectral theory to the temperature distribution and thermal radio emission of a half-space medium, are presented. Analytic closed-form expressions for the covariance functions of the temperature profile and the brightness temperature, represented as one-dimensional integrals of the spectrum of the surface temperature, have been obtained.

1. Introduction

The results of the simultaneous solution of radiative transfer and thermal conductivity equations, obtained in previous works [1,2], have been used to develop a stochastic theory of physical temperature and thermal radioemission of a medium considering the surface temperature as a random function of time [3]. This theory provides all the covariance functions and related statistical parameters as linear integrals of the surface temperature covariance function. The integral expressions obtained were of the convolution type, hence it was convenient to apply the spectral approach for the ensuing analysis. This paper extends the previous work by providing much simpler expressions for all second-order statistical parameters in the form of one-dimensional integrals. For the specific case of an exponential surface temperature covariance function explicit closed-form expressions for some parameters are obtained.

2. Problem formulation

Consider a homogeneous half-space $z \le 0$ with the following constant parameters: (i) thermal diffusion coefficient a^2 , and (ii) electromagnetic absorption coefficient γ . We allow that γ can depend on the wavelength λ). Two expressions [3] are used in the following analysis. The first of them determines the temperature profile inside the half-space as a function of depth and time for a temperature boundary condition for $T(0,t) = T_0(t)$:

$$T(z,t) = \int_0^\infty T_0(t-\tau)\frac{|z|}{\sqrt{4\pi a^2\tau^3}}exp(-\frac{z^2}{4a^2\tau})d\tau \ . \tag{1}$$

The second is the expression for brightness temperature of upward thermal radioemission as functional of surface temperature [2]:

$$T_B(t)=\int_0^\infty T_0(t-\tau)[\frac{\gamma a}{\sqrt{\pi\tau}} - (\gamma a)^2 erfc(\gamma a\sqrt{\tau})e^{(\gamma a)^2\tau}]d\tau. \tag{2}$$

Without lost of generality , we suppose that the surface reflection coefficient is zero.

Noting that both of the expressions (1) and (2) are equations of the convolution type ($y(t) = {}_0\int^\infty x(t-\tau)h_x(\tau)d\tau$) it follows that the Fourier transform of the kernel $h_x(\tau)$ determines the spectral transfer function $H_x(i\omega) = {}_{-\infty}\int^\infty h_x(\tau)\exp(-i\omega\tau)d\tau$, where $h_x(\tau) = 0$ for $\tau < 0$. From (1) and (2) one has respectively:

$$H_T(i\omega) = e^{-\frac{|z|}{a}\sqrt{i\omega}} \ , \tag{3}$$

$$H_{T_B}(i\omega) = \frac{1}{1+\sqrt{\frac{i\omega}{(\gamma a)^2}}} \ . \tag{4}$$

The modulus square of (3) and (4) are the power spectral transfer functions:

$$|H_T(i\omega)|^2 = e^{-\frac{|z|}{a}\sqrt{2\omega}}, \tag{5}$$

$$|H_{T_B}(i\omega)|^2 = \frac{1}{1+\frac{\sqrt{2\omega}}{\gamma a}+\frac{\omega}{(\gamma a)^2}}, \tag{6}$$

and will be used extensively, in the following analysis.

3. Derivation of statistical parameters

If the boundary condition for the temperature is a random stationary function with average value $\langle T_0\rangle$, mean square deviation σ_{T0}, and autocovariance function $B_{T_0T_0}(\tau)=\langle(T_0(t)-\langle T_0\rangle)(T_0(t+\tau)-\langle T_0\rangle)\rangle$ with corresponding spectral density $\Phi_{T0T0}(\omega) = \int_{-\infty}^{\infty} B_{T_0T_0}(\tau)\exp(-i\omega\tau)d\tau$, then the spectral densities of the physical temperature and radiobrightness can be expressed as:

$$\Phi_{TT}(\omega) = \left|H_T(i\omega)\right|^2 \Phi_{T_0T_0}(\omega) \tag{7}$$

$$\Phi_{T_BT_B}(\omega) = \left|H_{T_B}(i\omega)\right|^2 \Phi_{T_0T_0}(\omega). \tag{8}$$

The inverse Fourier transform for real processes of the formulas (7) and (8) determines the autocovariance functions for the temperature (at an arbitrary depth level z) $B_{TT}(\tau.z) = \langle(T(t,z)-\langle T\rangle)(T(t+\tau,z)-\langle T\rangle)\rangle$ and for the radiobrightness (at a given wavelength λ) $B_{T_BT_B}(\tau,\lambda) = \langle(T_B(t)-\langle T_B\rangle)(T_B(t+\tau)-\langle T_B\rangle)\rangle$:

$$B_{TT}(\tau,z) = \frac{1}{\pi}\int_0^{\infty} e^{-\frac{|z|}{a}\sqrt{2\omega}} \cos(\omega\tau)\Phi_{T0T0}(\omega)\, d\tau\ , \tag{9}$$

$$B_{T_BT_B}(\tau,\lambda) = \frac{1}{\pi}\int_0^{\infty} \frac{\cos(\omega\tau)\Phi_{T0T0}(\omega)}{1+\frac{\sqrt{2\omega}}{\gamma(\lambda)a}+\frac{\omega}{(\gamma(\lambda)a)^2}}\, d\tau\ . \tag{10}$$

Using autocovariance functions (9-10), general expressions obtained in [3] allow the determination of three basic covariance functions: (i) cross-level (for depth levels $z=z_1$ $z=z_2$) temperature covariance function $B_{T2T1}(z_2,z_1,\tau) = \langle(T(t,z_2)-\langle T(z_2)\rangle)(T(t+\tau,z_1)-\langle T(z_1)\rangle)\rangle$, (ii) the radiobrightness covariance function $B_{TB1TB2}(\lambda_1,\lambda_2,\tau)= \langle(T_B(t,\lambda_1)-\langle T_B(\lambda_1)\rangle)(T_B(t+\tau,\lambda_2)-\langle T_B(\lambda_2)\rangle)\rangle$, and (iii) the cross-covariance function $B_{TBT}(\lambda,z,\tau) = \langle(T_B(t,\lambda)-\langle T_B(\lambda)\rangle)(T(t+\tau,z)-\langle T(z)\rangle)\rangle$. Using these functions, it is possible to express all of the statistical parameters of the thermal and radio emission functions of the homogeneous medium. The property $B_{yx}(\tau) = B_{xy}(-\tau)$ also should be taken into account. For the mean values one has $\langle T(z)\rangle = \langle T_{0},\rangle$ $\quad\quad$ $\langle T_B\rangle = \langle T_0\rangle$.

In [3] the following expression for temperature covariance function was obtained:

$$B_{T2T1}(z_2,z_1,\tau) = \int_0^{\infty} B_{TT}(\tau-\tau',z_2)\,\frac{|z_1-z_2|}{2\sqrt{\pi}a}\,e^{-\frac{(z_1-z_2)^2}{4a^2\tau'}}\,\frac{d\tau'}{(\tau')^{3/2}}\ . \tag{11}$$

Substituting (9) in (11), changing the order of integration, and performing the internal integral over τ', obtain :

$$B_{T_2T_1}(z_2,z_1,\tau) = \int_0^{\infty} \frac{1}{\pi} exp\Big(-\frac{\sqrt{2\omega}}{a}\frac{|z_1|+|z_2|}{2}\Big)\, cos\Big(\omega\tau - \frac{\sqrt{2\omega}}{a}\frac{|z_1|-|z_2|}{2}\Big)\,\Phi_{T0T0}(\omega)\, d\omega\ . \tag{12}$$

The expression for the radiobrightness covariance function can similarly be written according to [3] as:

$$B_{T_{B1}T_{B2}}(\lambda_1,\lambda_2,\tau) = \frac{\gamma_2}{\gamma_1} B_{T_BT_B}(\tau,\lambda_1) + \tag{13}$$

$$+\Big(1-\frac{\gamma_2}{\gamma_1}\Big)\int_0^{\infty} B_{T_BT_B}(\tau-\tau',\lambda_1)\,\frac{\gamma_2 a}{\sqrt{\pi\tau'}}[1-\sqrt{\pi\tau'}(\gamma_2 a)\, e^{(\gamma_2 a)^2} erfc(\gamma_2 a\sqrt{\tau'})\, d\tau'$$

Substituting (10) in (13), changing the order of integration (as was made above), and performing the internal integral over τ' ,we have:

$$B_{T_{B1}T_{B2}}(\lambda_1,\lambda_2,\tau) = \int_0^{\infty} \frac{1}{\pi}\,\frac{cos(\omega\tau)\Big[1+\frac{\sqrt{2\omega}}{2a}\Big(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\Big)+\frac{\omega}{a^2\gamma_1\gamma_2}\Big] + sin(\omega\tau)\,\frac{\sqrt{2\omega}}{2a}\Big(\frac{1}{\gamma_2}-\frac{1}{\gamma_1}\Big)}{\Big(1+\frac{\sqrt{2\omega}}{\gamma_1 a}+\frac{\omega}{(\gamma_1 a)^2}\Big)\Big(1+\frac{\sqrt{2\omega}}{\gamma_2 a}+\frac{\omega}{(\gamma_2 a)^2}\Big)}\,\Phi_{T0T0}(\omega)\, d\omega \tag{14}$$

From the expression for the temperature-radiobrightness cross-covariance function [3] we have :

$$B_{T_BT}(\lambda,z,\tau) = \int_0^{\infty} B_{T_BT_B}(\tau-\tau',\lambda)\,\frac{|z|}{2\sqrt{\pi}a}\,e^{-\frac{z^2}{4a^2\tau'}}\,\frac{d\tau'}{(\tau')^{3/2}} + \frac{1}{\gamma(\lambda)a}\int_0^{\infty}\frac{\partial B_{T_BT_B}(\tau-\tau',\lambda)}{\partial\tau'}\,e^{-\frac{z^2}{4a^2\tau'}}\,\frac{d\tau'}{(\pi\tau')^{1/2}}, \tag{15}$$

Finally, substituting the autocovariance function (10) and making the same transformations as above, we obtain

(16)

$$B_{T_BT}(\lambda,z,\tau) = \int_0^\infty \frac{1}{\pi} \frac{\cos(\omega\tau - \frac{|z|}{a}\frac{\sqrt{2\omega}}{2}) - \frac{\sqrt{2\omega}}{2\gamma a}[\sin(\omega\tau - \frac{|z|}{a}\frac{\sqrt{2\omega}}{2}) - \cos(\omega\tau - \frac{|z|}{a}\frac{\sqrt{2\omega}}{2})]}{1 + \frac{\sqrt{2\omega}}{\gamma a} + \frac{\omega}{(\gamma a)^2}} e^{-\frac{|z|}{a}\frac{\sqrt{2\omega}}{2}} \Phi_{T0T0}(\omega)\, d\omega$$

The above expressions determine the basic statistical parameters of physical temperature radio emission of the half-space in the form of one-dimensional integrals instead of two-dimensional and three-dimensional integrals in common expressions in [3].

Specifically, values for the rms variances of temperature at level z and radiobrightness at wavelength λ are determined from (12) and (14), respectively, as $\sigma_T^2(z) = B_{T1T2}(z,z,0)$ and $\sigma_{TB}^2(\lambda) = B_{TB_1TB_2}(\lambda,\lambda,0)$. The autocovariance function of temperature at level z is determined from (12) as $B_{TT}(z,\tau) = B_{T1T2}(z,z,\tau)$, and the autocovariance function of radiobrightness at wavelength λ - from (14) as $B_{TBTB}(\lambda,\tau) = B_{TB1TB2}(\lambda,\lambda,\tau)$. It is further possible to determine all the correlation functions of temperature and radiobrightness from (12), (14), (16) as $R_{xz} = \frac{B_{xy}}{\sigma_x\sigma_y}$, i.e. $R_{T_1T_2}(z_1,z_2,\tau) = \frac{B_{T_1T_2}(z_1,z_2,\tau)}{\sigma_T(z_1)\sigma_T(z_2)}$, $R_{T_{B1}T_{B2}}(\lambda_1,\lambda_2,\tau) = \frac{B_{T_{B1}T_{B2}}(\lambda_1,\lambda_2,\tau)}{\sigma_{T_B}(\lambda_1)\sigma_{T_B}(\lambda_2)}$, $R_{T_BT}(\lambda,z,\tau) = \frac{R_{T_BT}(\lambda,z,\tau)}{\sigma_{T_B}(\lambda)\sigma_T(z)}$.

One can see that any of the above covariance functions can be expressed via time scale parameters as introduced in [2]. Specifically, the time of skin-depth heating is $\Gamma = 1/(\gamma a)^2$ where the skin-depth is $d = 1/\gamma$, and the time of heating of the layer with the depth z is $\Gamma_z = z^2/a^2$. In addition to these parameters, the covariance functions must depend on at least one time parameter of the surface temperature covariance function.

For an exponential surface temperature covariance function of the form :

$$B_{T0T0}(\tau) = \sigma^2 \exp(-\tau/\tau_0), \tag{17}$$

where there is a single time parameter τ_0 (the correlation time), the spectrum Φ_{T0T0} is :

$$\Phi_{T0T0}(\omega) = \frac{2\sigma^2}{\tau_0[\omega^2 + (\frac{1}{\tau_0})^2]}, \tag{18}$$

and the covariance functions (12), (14), (16) can be expressed using only three dimensionless parameters $r_Z = \sqrt{\frac{\Gamma_Z}{\tau_0}}$, $r = \sqrt{\frac{\Gamma}{\tau_0}}$, $r_\tau = \sqrt{\frac{\tau}{\tau_0}}$, along with the parameter σ^2:

$$B_{T_2T_1}(r_{Z_2}, r_{Z_1}, r_\tau) = \frac{8\sigma^2}{\pi}\int_0^\infty e^{-\frac{r_{Z1}+r_{Z2}}{2}\omega} cos(\frac{r_\tau^2}{2}\omega^2 - \frac{r_{Z1}-r_{Z2}}{2}\omega)\frac{d\omega}{\omega^4+4}, \tag{19}$$

$$B_{T_{B1}T_{B2}}(r_1, r_2, r_\tau) = \frac{32\sigma^2}{\pi}\int_0^\infty \frac{[\frac{r_1r_2}{2}\omega^2 + \frac{r_1+r_2}{2}\omega + 1]\, cos(\frac{r_\tau^2}{2}\omega^2) + \frac{r_2-r_1}{2}\omega\, sin(\frac{r_\tau^2}{2}\omega^2)}{(r_1^2\omega^2+2r_1\omega+2)(r_2^2\omega^2+2r_2\omega+2)}\frac{d\omega}{\omega^4+4}, \tag{20}$$

$$B_{T_BT}(r, r_z, r_\tau) = \frac{16\sigma^2}{\pi}\int_0^\infty e^{-\frac{r_Z}{2}\omega}\frac{cos\frac{r_\tau^2\omega^2 - r_z\omega}{2} - \frac{1}{2}r\omega[sin\frac{r_\tau^2\omega^2 - r_z\omega}{2} - cos\frac{r_\tau^2\omega^2 - r_z\omega}{2}]}{r^2\omega^2+2r\omega+2}\frac{d\omega}{\omega^4+4}, \tag{21}$$

It is possible to see from (19)-(21) that the autocovariance functions (which are also functions of τ_0) should depend only on two of the above dimensionless parameters, i.e. $B_{TT}(z,\tau,\tau_0) = B_{TT}(r_z, r_\tau)$, $B_{T_BT_B}(\lambda,\tau,\tau_0) = B_{T_BT_B}(r, r_\tau)$, and rms variances $\sigma^2_{TB}(\lambda,\tau_0) = \sigma^2_{TB}(r)$, $\sigma^2_T(z,\tau_0) = \sigma^2_T(r_Z)$ depend only on one of them. Those simple dependences allow the possibility to study these results in detail.

For the exponential covariance function (14) some statistical parameters can be expressed in explicit form. Particularly, we have :

$$\sigma^2_{T_B} = \sigma^2(\frac{4}{\pi}\frac{r^2 \ln r}{r^4-1} + \frac{1-r}{(r^2+1)(r+1)}), \tag{22}$$

$$B_{T_BT_0}(\tau) = \frac{\sigma^2}{1+r}e^{-\left|\frac{\tau}{\tau_0}\right|} = \frac{\sigma^2}{1+r}e^{-r_\tau^2} \quad \tau \geq 0, \tag{23}$$

$$= \sigma^2\{e^{-r_\tau^2} + e^{-r_\tau^2}\frac{r^2}{1+r^2}[\frac{r_\tau}{r}\,{}_1F_1(\frac{1}{2},\frac{3}{2},r_\tau^2) + e^{-[(\frac{r_\tau}{r})^2 + r_\tau^2]} erfc(\frac{r_\tau}{r}) - 1] -$$

$$- e^{r_\tau^2}\frac{r^2}{1-r^2}[\frac{1}{r}erfc(r_\tau^2) - e^{[(\frac{r_\tau}{r})^2 - r_\tau^2]} erfc(\frac{r_\tau}{r})]\}, \tau<0,$$

$$B_{TT_0}(\tau) = \sigma^2\, e^{-\left|\frac{z}{\Lambda}\right| - \frac{\tau}{\tau_0}} = \sigma^2\, e^{-(r_z + r_\tau^2)}, \quad \tau \geq 0 \quad , \qquad (24)$$

where $\Lambda = a\sqrt{\tau_0}$. The parameter Λ determines the depth scale of random temperature variations that penetrate into the medium. Perhaps the most interesting point is the fact that the correlation maximum is achieved with some time shift. This shift occurs because the covariance functions are not symmetric with respect to time shift, in the future and past. As it was mentioned in [3], from the physical point of view, the shift occurs because the temperature variations diffuse from the surface into the medium not instantly but by means of the process of thermal conductivity.

4. Numerical investigations

For the exponential covariance function (17) the numerical simulation has been carried out using the expressions (19-21) for all three primary covariance functions. The use of dimensionless parameters makes the results valid for a wide range of possible values of real parameters. The basic correlation features of physical temperature and radiobrightness are shown in the following figures.

In Fig.1, it is seen that the correlation function $R_{T_0T}(r_{z1} = 0, r_{z2} = r_z, r_\tau) = \dfrac{B_{T_1T_2}(0, r_z, r_\tau)}{\sigma_T(0)\sigma_T(r_z)}$ between the surface temperature and the temperature at depth level z has a dependence on the dimensionless parameters r_z and r_τ, representing the depth and time shift, respectively. One can see that this function, which is symmetric relative to the point $r_\tau = 0$ (i.e. $\tau = 0$) on the surface, gradually looses its symmetry as the depth increases, and that its maximum, gradually decreasing, shifts into the future.

In Fig.2, the correlation function $R_{T_1T_2}(r_{z_1} = 2, r_{z_2}, r_\tau) = \dfrac{B_{T_1T_2}(r_{z_1} = 2, r_{z_2}, \tau)}{\sigma_T(r_{z_1} = 2)\sigma_T(r_{z_2})}$ between the temperature at depth level z_2 and the temperature at z_1 are shown for a fixed value $r_{z1} = 2$ (i.e., $\Gamma_{z1} = 4\tau_0$, representing the case $\Gamma_{z1} >> \tau_0$) and as a function of r_{Z2} and r_τ. This function is symmetric relative to the time shift at level $r_{Z2} = r_{Z1}$, but is more broad than (17). By shifting z_2 to the surface the correlation maximum shifts, decreasing into the past, and, visa versa, by moving away from the surface the maximum shifts in the future.

The cross-correlation function $R_{T_BT}(r=1,r_z,r_\tau)=\dfrac{R_{T_BT}(r=1,r_z,r_\tau)}{\sigma_{T_B}(r=1)\sigma_T(r_z)}$ between the radiobrightness at fixed value $r = 1$ (i.e., $\Gamma = \tau_0$) and the temperature at level z is shown in Fig.3. This function achieves its maximum at the depth level z, at which the dimensionless parameter r_z is close to value r . By shifting z from this level to the surface, the correlation maximum shifts in the past, and by moving z away from the surface, it shifts in the future.

In fig.4, the correlation function of radiobrightness $R_{T_{B1}T_{B2}}(r_1=1,r_2,r_\tau)=\dfrac{B_{T_{B1}T_{B2}}(r_1=1,r_2,r_\tau)}{\sigma_{T_B}(r_1=1)\sigma_{T_B}(r_2)}$ is shown. At the wavelength λ_1 the dimensionless parameter ($r_1 = 1$, i.e. $\Gamma_1 = \tau_0$) is fixed, and one can see the dependence of the correlation function on the parameter r_2 (for the second wavelength λ_2) and on time parameter r_τ. The correlation function achieves its maximum at $r_2 = r_1$, and very slowly diminishes, gradually shifting in the future with the increasing of r_2 , or to the past for $r_2 < r_1$.

5. Conclusion

Using the spectral approach, the development of the correlation theory of physical temperature and thermal emission for a homogeneous half-space as published in [3] was completed. All of the second-order statistical parameters of the temperature and radio emission were expressed as one-dimensional integrals of the surface temperature spectrum using kernels which depend on the medium parameters of absorption and thermal conductivity. For the exponential surface temperature covariance function some results were obtained in explicit closed-form and numerical simulations were carried out.

The results of the investigation should be applicable to real media (atmosphere, soils, oceans), since the case of a homogeneous medium is common. It might be possible to investigate an inhomogeneous case on the basis of a numerical analysis in the framework of the general approach described in [3].

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References

1. K.P.Gaikovich, A.N.Reznik and R.V.Troitskii, "Microwave subsurface profile thermometry," Dig.IGARSS'91, Univ. Technol., Espoo, Finland, vol.3, pp.1195-1198, 1991.
2. K.P.Gaikovich, "Simultaneous solution of emission transfer and thermal conductivity equations in the problems of atmosphere and subsurface radiothermometry," IEEE Trans.Geosci.Remote Sens., vol.32, pp.885-889, 1994.
3. K.P.Gaikovich, "Stochastic theory of temperature distribution and thermal emission of half-space with random time-dependent surface temperature," IEEE Trans.Geosci. Remote Sens., vol.34, No.2, pp.582-587, 1996.

Figure captures

Fig.1. The temperature correlation function R_{T0T} between temperature and its surface value with dependence on the dimensionless parameters of depth r_z and time r_τ.

Fif.2. The temperature correlation function R_{T1T2} between the temperature at fixed depth ($r_{z1} = 2$) and the temperature at another level (determined by parameter r_{z2}) with dependence on r_{z2} and r_τ.

Fig.3. The cross-correlation function R_{T_BT} between the radiobrightness with the parameter $r = 1$ and the temperature at the depth level z, as determined by the parameter r_z, with dependence on r_z and r_τ.

Fig.4. The radiobrightness correlation function R_{TB1TB2} between the radiobrightness at two different wavelengths, one of which is fixed ($r_1 = 1$) with dependence on r_2 and r_τ.