Stochastic Theory of Temperature Distribution and Thermal Emission of Half-Space with Random Time-Dependent Surface Temperature

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***Absrtact*-On the basis of results of simultaneous solution of thermal emission transfer and thermal conductivity equations the stochastic theory of temperature distribution and thermal radioemission of medium (half-space) has been developed. Expressions for covariance functions of temperature profile and brightness temperature as functions of statistical parameters of half-space surface temperature which was considered as a random function of time have been obtained.**

I. Introduction

In previous works [1-2] the theory of radioemission in medium (half-space) with temperature distribution which depends on boundary conditions (surface temperature or heat flux) dynamics has been developed. On the base of simultaneous solution of emission transfer and thermal conductivity equations it appeared possible to obtain expressions for brightness temperature of radioemission as integrals of boundary conditions evolution [1], and next, to make inversion of these expressions and to obtain formulas for boundary conditions and temperature distribution (profile) of the medium as integrals of brightness temperature evolution [2]. Since, it appeared possible to obtain a correct solution of the problem of one-wavelength temperature profile retrieval. These results have been applied for radiometer investigations of diurnal heat dynamics in soil (by brightness temperature measurements at wavelengths 0.8 and 3 cm), and also for investigations of atmosphere boundary layer (by brightness temperatures at wavelength 0.5 cm in the oxygen line center).

There are other methods for temperature profile retrieval by spectrum or by angle dependence of brightness temperatures measured at the same time. But to use these methods it is necessary to solve incorrect integral Fredholm equations of the 1-st kind. It is impossible without using additional *a priori* information about unknown solution. This information can be the information about generous function properties such as smoothness, differentiability, or belonging to one of compact sets (Tikhonov's methods) [1,3]. It can be also a statistical information. In this case covariance functions and other statistical parameters of radioemission and temperature profile are in use. These functions and parameters are determined from radiosondes statistics of meteorological network [4-6]. For the cases of atmosphere boundary layer or subsurface soil sounding it is difficult (or, more often, impossible) to take statistics, and in these cases Tikhonov's methods have been applied [1,3]. But the results of simultaneous solution of emission transfer and thermal conductivity equations [1-2] can be used not only as a new retrieval method, but also to develop the stochastic theory of the medium, if one considers surface temperature as a random function of time. This theory gives necessary covariance functions and statistical parameters.

II. Problem Formulation

Let us consider the homogeneous half-space $z \le 0$ with the constant parameters: thermal diffusivity coefficient a^2 and absorption (of thermal radioemission) coefficient γ. If we have boundary condition for temperature $T(0,t) = T_0(t)$, then the temperature distribution inside the half-space can be determined from thermal conductivity equation as a function of depth and time as follows:

$$T(z,t) = \int_{-\infty}^{t} T_0(\tau) \frac{-z}{\sqrt{4\pi a^2 (t-\tau)^3}} exp\left(-\frac{z^2}{4a^2(t-\tau)}\right) d\tau \quad .$$

(1)

The brightness temperature of upward thermal radioemission at wavelength λ is determined from emission transfer equation:

$$T_B(\lambda,t) = \int_{-\infty}^{0} T(z,t)\, \gamma(\lambda)\, e^{\gamma(\lambda) z}\, dz \quad .$$

(2)

(for briefer formulation it is supposed that the reflection coefficient is zero).

The simultaneous solution of (2) and (3) gives the expression for brightness temperature as functional of surface temperature [2]:

$$T_B(t) = \int_{-\infty}^{t} T_0(\tau) \left[\frac{\gamma a}{\sqrt{\pi (t-\tau)}} - (\gamma a)^2 erfc(\gamma a \sqrt{t-\tau})\, e^{(\gamma a)^2 (t-\tau)} \right] d\tau \ .$$

(3)

For the case of inhomogeneous medium the correspondent expression also have been obtained in [2]:

$$T_B(t) = \int_{-\infty}^{t} T_0(\tau) \frac{\partial}{\partial t} T_B^{1}(t-\tau) d\tau \ , \qquad (4)$$

where $T_B^{1}(t-\tau)$ - brightness temperature response by unity change of surface temperature:

$$T^1(0,t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

The solution of (3) as Volterra's equation of the 1-st kind with the variable upper limit obtained in [2] can be expressed as

$$T_0(t) = T_B(t) + \frac{1}{\gamma a} \int_{-\infty}^{t} T_B'(\tau) \frac{d\tau}{\sqrt{\pi(t-\tau)}} =$$
$$= T_B(t) + \frac{1}{\gamma a} \int_{-\infty}^{t} (T_B(t) - T_B(\tau)) \frac{d\tau}{\sqrt{\pi(t-\tau)^3}} \ .$$

(5)

The substitution (5) in (1) gives the solution of the problem of one-wavelength radiothermometry for homogeneous half-space[2]:

(6)

$$T(z,t) = \int_{-\infty}^{t} T_B(\tau)(-z) e^{-\frac{z^2}{4a^2(t-\tau)}} \frac{d\tau}{\sqrt{4\pi a^2(t-\tau)^3}} + \frac{1}{\gamma a} \int_{-\infty}^{t} T_B'(\tau) e^{-\frac{z^2}{4a^2(t-\tau)}} \frac{d\tau}{\sqrt{\pi(t-\tau)}}$$

Performing the integration of the second term in (6) by parts it is possible to obtain the expression which permits the determination of the temperature profile by brightness temperature evolution [2]:

$$T(z,t)=\int_{-\infty}^{t} T_B(\tau) e^{-\frac{z^2}{4a^2(t-\tau)}} \left[\frac{1}{\gamma}\left(\frac{z^2}{2a^2(t-\tau)}-1\right)-z\right] \frac{d\tau}{\sqrt{4\pi a^2(t-\tau)^3}} \quad . \tag{7}$$

The above expression is valid for all values of z with the exception of z = 0 where it is impossible to perform the integration by parts in (6). This important property has not been discovered in the previous works.

This theory gives also full information about heat flux dynamics $J(z,t)$ in the medium. The correspondent expression can be obtained from (6):

$$J(z,t)=-k\frac{\partial T}{\partial z}=\int_{-\infty}^{t} T_B(\tau)\, k e^{-\frac{z^2}{4a^2(t-\tau)}} \left(1-\frac{z^2}{2a^2(t-\tau)}\right)\frac{d\tau}{\sqrt{4a^2(t-\tau)^3}} -$$

$$-\int_{-\infty}^{t} T_B'(\tau)\frac{kz}{\gamma a^2} e^{-\frac{z^2}{4a^2(t-\tau)}} \frac{d\tau}{\sqrt{4a^2(t-\tau)^3}}.$$

In this expression k is the thermal conductivity coefficient.

On the base of this theory it is possible to obtain one more useful result - the formula which expresses the brightness temperature at one wavelength as a functional of evolution of brightness temperature at another wavelength. To take this formula it is necessary to substitute the temperature profile (6) expressed as function of evolution of T_{B1} at λ_1 in (3) for T_{B2} at λ_2. Changing the integration sequence and taking internal integral by z one has the final expression:

$$T_{B2}(t)=\frac{\gamma_2}{\gamma_1}T_{B1}(t)+\int_{-\infty}^{t} T_{B1}(\tau)\left(1-\frac{\gamma_2}{\gamma_1}\right)\left\{\gamma_2 a\left[\frac{1}{\sqrt{\pi(t-\tau)}}-\gamma_2 a\, e^{(\gamma_2 a)^2(t-\tau)}\operatorname{erfc}\left(\gamma_2 a\sqrt{t-\tau}\right)\right]\right\}d\tau \tag{8}$$

where γ_1, γ_2 - absorption coefficients at wavelengths λ_1 and λ_2 respectively. It should be mentioned that it is impossible to use for this purpose the expression (7) instead of (6) because the formula (7) is invalid in one point z = 0, (integrand expression in (7) has the singularity in this point).

From (8) at $\gamma_1 = \gamma_2$ one has the obvious result $T_{B2} = T_{B1}$. In the case, when $\gamma_2 << \gamma_1$ the first term in (8) disappeares, and one has the formula similar to (3) where instead of surface temperature appears the brightness temperature T_{B1}. It is an evident result because for skin-depth $d = 1/\gamma$ one has the relation $d_1 << d_2$. It is well known that the main contribution in the value of brightness temperatute in (2) gives the temperature profile in the layer from surface level to level $|z| = d$, hence the brightness temperature T_{B1} in this case plays a role of surface temperature. It is possible to use the expression (8) also for determination of medium parameters by simultaneous measurements of thermal radioemission at two or more wavelengths.

III. Stochastic theory of half-space

Now, let us consider the boundary condition for the temperature as a random stationary function with the middle value $\langle T_0 \rangle$, mean square deviation σ_{T_0} and autocovariance function $B_{T_0T_0}(\tau) = \langle (T_0(t)-\langle T_0\rangle)(T_0(t+\tau)-\langle T_0\rangle)\rangle$ which for evaluations will be used in the form:

$$B_{T_0T_0}(\tau) = \sigma^2_{T_0} \, exp(-\left|\frac{\tau}{\tau_0}\right|) \tag{9}$$

where τ_0 is the correlation time. The random temperature variations on the surface for cases of atmosphere and soil can be related with mesoscale and large scale weather processes. For water and wet soils surface temperature variations can be caused also by evaporation variations related with wind speed variations. So, in each case it is necessary to use for evaluations the properly choosen value of τ_0 .

The purpose of the following analysis is determination of covariance functions of temperature distribution and brightness temperature using medium parameters and statistical parameters of the surface temperature. It is clear that for mean values $\langle T(z)\rangle = = \langle T_0\rangle$, $\langle T_B\rangle = \langle T_0\rangle$, because of the unity normalization of correspondent integral expressions.

If the surface temperature is a random function then using the fact that these integral expressions are linear, it is possible to develop the stochastic theory for the random components of temperature distribution and thermal radioemission of the medium on the basis of known approach in the theory of stationary random processes for linear systems which leads to Wiener-Li expressions. The following results determine the stochastic parameters of temperature distribution and thermal radioemission of the medium using known stochastic parameters of the surface temperature. These formulas can be easy obtained from above expressions by means of changing the integration and averaging sequence along with variables change $\tau' = t\text{-}\tau$. The results are given in the form which is valid both for positive and negative half-space. The property of covariance functions $B_{yx}(-\tau)=B_{xy}(\tau)$ is also in use.

So, from (1) for covariance function between surface temperature and temperature $T(z)$ at level z we have

$$B_{T_0T}(\tau) = \int_0^\infty B_{T_0T_0}(\tau-\tau')K(\tau')d\tau' = \int_0^\infty B_{T_0T_0}(\tau-\tau')\frac{|z|}{2\sqrt{\pi a}}e^{-\frac{z^2}{4a^2\tau'}}\frac{d\tau'}{(\tau')^{3/2}} \qquad (10)$$

where $K(\tau')$ is the kernel of the integral in (1). From this we have the expression for mean square variance

$$\sigma_T^2(z) = \int_0^\infty B_{T_0T}(\tau)K(\tau)d\tau\,, = \int_0^\infty\int_0^\infty B_{T_0T_0}(\tau-\tau')K(\tau')K(\tau)d\tau d\tau' =$$
$$= \int_0^\infty\int_0^\infty B_{T_0T_0}(\tau-\tau')\frac{z^2}{4\pi a^2}e^{-\frac{z^2}{4a^2}\left(\frac{1}{\tau'}+\frac{1}{\tau}\right)}\frac{d\tau d\tau'}{(\tau\tau')^{3/2}} \quad , \qquad (11)$$

and for autocovariance function of temperature at level z

$$B_{TT}(\tau) = \int_0^\infty B_{T_0T}(\tau'-\tau)K(\tau')d\tau' = \int_0^\infty\int_0^\infty B_{T_0T_0}(\tau'-\tau''-\tau)K(\tau')K(\tau'')d\tau'd\tau'' =$$
$$= \int_0^\infty\int_0^\infty B_{T_0T_0}(\tau'-\tau''-\tau)\frac{z^2}{4\pi a^2}e^{-\frac{z^2}{4a^2}\left(\frac{1}{\tau'}+\frac{1}{\tau''}\right)}\frac{d\tau'd\tau''}{(\tau'\tau'')^{3/2}} \quad , \qquad (12)$$

and also for interlevel covariance function between temperature variances T_1 at the level z_1 and T_2 at the level z_2 (in that case T_2 in the kernel of the integral in (1) plays a role of surface temperature):

$$B_{T_2T_1}(z_2,z_1,\tau) = \int_0^\infty B_{T_2T_2}(\tau-\tau')K(\tau')d\tau' =$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty B_{T_0T_0}(\tau'-\tau''+\tau'''-\tau)\frac{z_2^2|z_1-z_2|}{8\pi^{3/2}a^3}e^{-\frac{z_2^2}{4a^2}\left(\frac{1}{\tau'}+\frac{1}{\tau''}\right)-\frac{(z_1-z_2)^2}{4a^2}\frac{1}{\tau'''}}\frac{d\tau'd\tau''d\tau'''}{(\tau'\tau''\tau''')^{3/2}} \ . \qquad (13)$$

The covariance function between surface temperature of the medium and brightness temperature of its radioemission can be expressed as

$$B_{T_0T_B}(\tau) = \int_0^\infty B_{T_0T_0}(\tau-\tau')K_1(\tau')d\tau' =$$
$$= \int_0^\infty B_{T_0T_0}(\tau-\tau')\frac{\gamma a}{\sqrt{\pi\tau'}}[1-\sqrt{\pi}(\gamma a)\sqrt{\tau'}e^{(\gamma a)^2\tau'}erfc(\gamma a\sqrt{\tau'})]d\tau' \qquad (14)$$

where K_1 is the integral kernel in (3) or in (4). The formula (14) leads to the expression for mean square variance

$$\sigma^2_{T_B} = \int_0^\infty B_{T_0T_B}(\tau)K_1(\tau)d\tau = \int_0^\infty\int_0^\infty B_{T_0T_B}(\tau-\tau')K_1(\tau')K_1(\tau)d\tau d\tau' = \qquad (15)$$
$$\int_0^\infty\int_0^\infty B_{T_0T_0}(\tau-\tau')\frac{(\gamma a)^2}{\pi\sqrt{\tau\tau'}}[1-\sqrt{\pi}(\gamma a)\sqrt{\tau'}e^{(\gamma a)^2\tau'}erfc(\gamma a\sqrt{\tau'})][1-\sqrt{\pi}(\gamma a)\sqrt{\tau}e^{(\gamma a)^2\tau}erfc(\gamma a\sqrt{\tau})]d\tau d\tau'$$

and for brightness temperature autocovariance function

$$B_{T_BT_B}(\tau) = \int_0^\infty B_{T_BT_0}(\tau-\tau')K_1(\tau')d\tau' = \int_0^\infty\int_0^\infty B_{T_0T_0}(\tau'-\tau''-\tau)K(\tau')K(\tau'')d\tau' d\tau'' = \qquad (16)$$
$$\int_0^\infty\int_0^\infty B_{T_0T_0}(\tau'-\tau''-\tau)\frac{(\gamma a)^2}{\pi\sqrt{\tau'\tau''}}[1-\sqrt{\pi}(\gamma a)\sqrt{\tau'}e^{(\gamma a)^2\tau'}erfc(\gamma a\sqrt{\tau'})][1-\sqrt{\pi}(\gamma a)\sqrt{\tau''}e^{(\gamma a)^2\tau''}erfc(\gamma a\sqrt{\tau''})]d\tau' d\tau''$$

It is possible to use (16) to obtain the expression for covariance function between brightness temperatures T_{B1} and T_{B2} at two different wavelengths λ_1 and λ_2 from (8)

$$B_{T_{B_1}T_{B_2}}(\lambda_1,\lambda_2,\tau) = \int_0^\infty B_{T_{B_1}T_{B_1}}(\tau-\tau')K_1(\tau')d\tau' =$$
$$= \frac{\gamma_2}{\gamma_1}B_{T_{B_1}T_{B_1}}(\tau) + (1-\frac{\gamma_2}{\gamma_1})\int_0^\infty B_{T_{B_1}T_{B_1}}(\tau-\tau')\frac{\gamma_1 a}{\sqrt{\pi\tau'}}[1-\sqrt{\pi}(\gamma_1 a)\sqrt{\tau'}e^{(\gamma_1 a)^2\tau'}erfc(\gamma_1 a\sqrt{\tau'})]d\tau' \qquad (17)$$

and also to obtain the formula for covariance function between brightness temperature and temperature at the level z from (6) and (7):

$$B_{T_BT}(\tau) = \int_0^\infty B_{T_BT_B}(\tau-\tau')\frac{|z|}{2\sqrt{\pi}a}e^{-\frac{z^2}{4a^2\tau'}}\frac{d\tau'}{(\tau')^{3/2}} + \frac{1}{\gamma a}\int_0^\infty \frac{\partial B_{T_BT_B}}{\partial\tau'}(\tau-\tau')e^{-\frac{z^2}{4a^2\tau'}}\frac{d\tau'}{\sqrt{\pi}(\tau')^{1/2}} =$$

$$= \int_0^\infty B_{T_B T_B}(\tau - \tau') \frac{|z|}{2\sqrt{\pi}a} \left[\frac{1}{\gamma}\left(\frac{z^2}{2a^2\tau'} - 1\right) - z \right] e^{-\frac{z^2}{4a^2\tau'}} \frac{d\tau'}{(\tau')^{3/2}} \ , \quad (|z| > 0) \ . \tag{18}$$

Of course, it is easy to obtain similar expressions for the heat flow in the medium.

On the base of this theory it is possible, in particular, to carry out a sensitivity study of the linkage between the time history and the thermal structure. Since the expressions (6), (7) are linear, it is possible to consider $B_{T_B T_B}$ in (18) as error covariance matrix $B_{\delta T_B \delta T_B}$, and for mean square error $(\delta T(z))^2$ of temperature profile determination one has, using (18),

$$(\delta T(z))^2 = \int_0^\infty \int_0^\infty B_{\delta T_B \delta T_B}(\tau - \tau') \frac{1}{4\pi a^2} \left[\frac{1}{\gamma}\left(\frac{z^2}{2a^2\tau} - 1\right) - z \right] \left[\frac{1}{\gamma}\left(\frac{z^2}{2a^2\tau'} - 1\right) - z \right] e^{-\frac{z^2}{4a^2}\left(\frac{1}{\tau}+\frac{1}{\tau'}\right)} \frac{d\tau' d\tau}{(\tau\tau')^{3/2}} \ .$$

If brightness error is time-independent, i.e. $\delta T_B = const$, then $\delta T(z) = \delta T_B$.

IV. Regression analysis and evaluations for atmosphere and subsoil radiothermometry

The expressions obtained are of great interest from the point of view of widely applied statistical methods of temperature profile retrieval by thermal radioemission or by surface temperature [4-6] because it appears possible now to use these expressions also for the cases of atmosphere boundary layer and subsurface soil sounding. The developed theory permits to determine necessary stochastic parameters. Moreover, the obtained results make clear the physical significance of these parameters. The expressions obtained give the possibility to make predictions in the future ($\tau > 0$), in the past ($\tau \le 0$), and simultaneously ($\tau = 0$).

It is easy to use the expression (10) for regression estimation of temperature profile at time t by surface temperature measured at time t-τ :

$$T(z,t) = \langle T_0 \rangle + \frac{B_{T_0 T}(\tau)}{\sigma_{T_0}^2} (T_0(t-\tau) - \langle T_0 \rangle) \quad . \tag{19}$$

Similarly, it is possible to estimate the temperature profile by brightness temperature, using (18):

$$T(z,t) = \langle T_0 \rangle + \frac{B_{T_B T}(\tau)}{\sigma_{T_B}{}^2}(T_B(t-\tau) - \langle T_0 \rangle) \qquad (20)$$

or to use more complicated methods of multivariate regression or statistical regularization [4-6]. It is clear that all other above mentioned covariance functions can be applied similarly for estimations of correspondent values. Thus, using (13), one can estimate the temperature at the level $z = z_2$ by the temperature at $z = z_1$ and, using (17), estimate the brightness temperature at one wavelength by brightness temperature at another wavelength. On the basis of (12) and (16) it is possible to make predictions of the temperature at given level and the brightness temperature at given wavelength in the future and in the past by their present values.

It is known that the mean square error $\sigma_{y/x}$ of regression estimation y by x depends on correlation coefficient $R_{xy} = \frac{B_{xy}}{\sigma_x \sigma_y}$, and can be calculated from presented above expression as

$$\sigma_{y/x}^2 = \sigma_y^2(1 - R_{xy}^2) \quad . \qquad (21)$$

From expressions for correlation functions one can see that these functions are not symmetric relative to the point $\tau=0$,and, moreover, it is not a point of their maxima, i.e. the prediction in the future is not symmetric relative to the prediction in the past. The prediction of the profile by simultaneous values of surface temperature or brightness temperature known as "optimal extrapolation" is not optimal in reality. One can see from (21) that the optimal estimation is the estimation of value to be predicted at time t by value of the predictor at such a time (t - τ_m) that for τ_m functions $R_{xy}(\tau_m)$ and, hence, $B_{xy}(\tau_m)$ will have their maxima. The condition from which the value of τ_m can be determined is obviously $\frac{dB(\tau)}{d\tau} = 0$, and the correspondent equations can be easy obtained from expressions for covariance functions. Concretely, for the temperature profile prediction, for each value of z there is the correspondent value $\tau_m(z)$, and the optimal extrapolation can be achieved not by simultaneous value of surface temperature, but by its value which was at time $t - \tau_m(z)$ in the past. It is obvious that value of τ_m increase with increasing of z, i.e. $\tau_m(z)$ is a monotonically increasing function of z. This property of regression estimation is obvious from the physical point of view because the surface temperature disturbance can influence the temperature of deep layers not at the same time but only by means of heat conductivity, with the delay which increases with depth.

It is possible to suppose that in real atmosphere (especially in the boundary layer) in spite of the fact that the real conditions are not in full correspondence with assumptions of this theory, the temperature correlation maximum must be also in the past.

For exponential covariance function (9) some simple results can be obtained. The expression (10) yields

$$B_{T_0T}(\tau) = \; = \int_0^\infty \frac{\sigma_{T_0}^2 |z|}{2\sqrt{\pi}a} e^{-\frac{z^2}{4a^2\tau} - \frac{|\tau-\tau'|}{\tau_0}} \frac{d\tau'}{(\tau')^{3/2}} \; = \tag{22}$$

$$= \sigma_{T_0}^2 e^{-\frac{|z|}{a\sqrt{\tau_0}} - \left|\frac{\tau}{\tau_0}\right|}, \qquad \tau \le 0,$$

$$= \frac{\sigma_{T_0}^2 |z|}{2\sqrt{\pi}a}\left[\int_0^\tau e^{-\frac{z^2}{4a^2\tau} + \frac{\tau'}{\tau_0}} \frac{d\tau'}{(\tau')^{3/2}} \times e^{-\frac{\tau}{\tau_0}} + \int_\tau^\infty e^{-\frac{z^2}{4a^2\tau} - \frac{\tau'}{\tau_0}} \frac{d\tau'}{(\tau')^{3/2}} \times e^{\frac{\tau}{\tau_0}}\right], \tau > 0$$

In particular,

$$B_{T_0T}(0) = \sigma_{T_0}^2 e^{-\frac{|z|}{a\sqrt{\tau_0}}} \quad . \tag{23}$$

One can see that there is a correlation depth $\Lambda = a\sqrt{\tau_0}$ which can be considered as one of definitions for atmosphere boundary layer (as a typical height scale of layer where there is appreciable influence of surface temperature variations on temperature profile variations). In this case the equation for the determination of optimal extrapolation time shift $\tau_m(z)$ can be written as follows:

$$\int_\tau^\infty e^{-\frac{z^2}{4a^2\tau} - \frac{\tau'}{\tau_0}} \frac{d\tau'}{(\tau')^{3/2}} \times e^{\frac{\tau}{\tau_0}} - \int_0^\tau e^{-\frac{z^2}{4a^2\tau} + \frac{\tau'}{\tau_0}} \frac{d\tau'}{(\tau')^{3/2}} \times e^{-\frac{\tau}{\tau_0}} = 0 \quad . \tag{24}$$

From (14) it is possible to obtain

$$B_{T_0T_B}(\tau) = \int_0^\infty \frac{\sigma_{T_0}^2 \gamma a}{\sqrt{\pi\tau'}} [1 - \sqrt{\pi}(\gamma a)\sqrt{\tau'} e^{(\gamma a)^2 \tau'} erfc(\gamma a\sqrt{\tau'})] e^{-\frac{|\tau-\tau'|}{\tau_0}} d\tau' \quad . \tag{25}$$

After integration of (25) one has:

$$B_{T_0T_B}(\tau) = \sigma^2_{T_0} \frac{\sqrt{\frac{\tau_0}{\Gamma}}}{1+\sqrt{\frac{\tau_0}{\Gamma}}} e^{-\left|\frac{\tau}{\tau_0}\right|} \;, \; \tau \le 0 \;, \tag{26}$$

$$= \sigma^2_{T_0}\{e^{-\frac{\tau}{\tau_0}} + \frac{1}{\tau_0} e^{-\frac{\tau}{\tau_0}} \frac{1}{[(\gamma a)^2 + \frac{1}{\tau_0}]} [\gamma a \sqrt{\tau}\, {}_1F_1(\frac{1}{2}, \frac{3}{2}, \frac{\tau}{\tau_0}) + e^{-[(\gamma a)^2 + \frac{1}{\tau_0}]\tau} erfc(\gamma a\sqrt{\tau}) - 1] -$$

$$- \frac{1}{\tau_0} e^{\frac{\tau}{\tau_0}} \frac{1}{[(\gamma a)^2 - \frac{1}{\tau_0}]} [\gamma a \sqrt{\tau_0}\, erfc(\frac{\tau}{\tau_0}) - e^{[(\gamma a)^2 - \frac{1}{\tau_0}]\tau} erfc(\gamma a\sqrt{\tau})]\} \;, \tau > 0$$

where ${}_1F_1$ is the singular hypergeometric function, $\Gamma = 1/(\gamma a)^2$ is time parameter which determines the heating of the medium at the skin depth $d=1/\gamma$ (because the evolution of $T_0(\tau)$ at $\tau << t$-Γ gives the negligible contribution in the value of $T_B(t)$ in (3)). The expression (26) yields:

$$B_{T_0T_B}(0) = \sigma^2_{T_0} \frac{\sqrt{\frac{\tau_0}{\Gamma}}}{1+\sqrt{\frac{\tau_0}{\Gamma}}} = \sigma^2_{T_0} \frac{\Lambda}{d+\Lambda} \;, \tag{27}$$

where the obvious relation $\sqrt{\frac{\tau_0}{\Gamma}} = \frac{\Lambda}{d}$ is used. One can see that the correlation of the brightness temperature of the medium with the surface temperature is determined by relation of surface temperature correlation time and time of skin-depth heating. At $\tau_0/\Gamma >> 1$ $B_{T_0T_B}(0) = \sigma^2_{T_0}$, and at $\tau_0/\Gamma << 1$ $B_{T_0T_B}(0) = 0$. This result is absolutely clear. If the medium is able to change its temperature at skin-depth during the correlation time of surface temperature, the brightness temperature will be completely correlated with the surface temperature. Otherwise, the variations of these values will be uncorrelated.

Considering the question about the possibility to apply the developed theory to atmosphere and soil investigations it is important to take into account the specific conditions of the case to be studied. In particular, for atmosphere a^2 is turbulent thermal diffusivity coefficient whereas for soil it is molecular one. In the case of soil (for homogeneous one especially) it is possible to expect that random components of temperature distribution and radioemission, which are superimposed on periodic diurnal and seasonal components, can be properly described by this theory. It should be mentioned, however, that if the constitutive properties have the strong temperature dependence (for example, near freezing), the problem becomes nonlinear and this theory is invalid.

In the case of atmosphere the random component of temperature profile can be formed not only by thermal conductivity but also by advection, condensation and evaporation of water, by transfer and absorption of infrared emission. The atmosphere is very often an inhomogeneous medium (especially for a^2) and its parameters can be time-dependent. If the typical time scale of this dependence is less than τ_0, the theory is not applicable. In the atmosphere boundary layer, however, under certain conditions and for measurements in the strong absorption lines, for example, in oxygen line at frequency 60 GHz where the absorption coefficient is a constant value, this stochastic theory can be applied as well as the initial formulas in [2]. But in any case the random component of temperature distribution connected with the thermal diffusivity plays an important role in the atmosphere, and it is possible to estimate its relative contribution in various cases on the basis of comparison of theoretical and empirical covariance functions.

Although the application of the theory to concrete cases is beyond the scope of this paper, for better understanding of typical time and height scales in above formulas some estimations of parameters for atmosphere and soil are presented. Large scale weather correlation time for surface temperature is typically about $\tau_0 \cong 2.6{\cdot}10^5$ s (3 days). It is clear that the value of a^2 in the atmosphere can be changed during this time, and in these cases the above theory will be invalid for large scale weather variations.

During the radiometer investigations of temperature dynamics of atmosphere boundary layer and soil presented in [2] the media parameters had the following values. For soil: $a^2 = 1.0{\cdot}10^{-3}$ cm^2/s, $d=1/\gamma \cong 1.0{\cdot}\lambda$, $\Gamma = 1/(\gamma a)^2$ was from 10 minutes at $\lambda = 0.8$ cm to 50 hr at $\lambda = 13$ cm, correlation depth in (20) $\Lambda = a\sqrt{\tau_0} \cong 16$ cm. One can see that at shorter wavelengths $\tau_0/\Gamma >> 1$ and the correlation between surface temperature and brightness temperature must be high. For atmosphere at frequency 60 GHz: $a^2 = 7.0{\cdot}10^3$ cm^2/s, $d=3.0{\cdot}10^4 \sin\theta$ cm (θ - elevation angle), the time parameter Γ was from 16 minutes for measurements at elevation angle 5^o to 35 hr for measurements in zenith direction, the correlation height $\Lambda \cong 430$ m.

In various natural conditions values of these parameters can range wider. For underground $a^2 = 10^{-3} - 10^{-2}$ cm^2/s, $d = 0.1\text{-}15\lambda$, Γ can be from 0.1 s for water surface in millimeter wavelengths up to years for ice at decimeter wavelengths, correlation depth $\Lambda = 15\text{-}60$ cm. For atmosphere $a^2 = 10^3 - 10^6$ cm^2/s , $d=3.0{\cdot}10^4 \sin\theta$ cm (at frequency 60 GHz), parameter Γ can be from 1 min at elevation angle 5^{o} up to 10 days for zenith direction. Typical correlation height scale (boundary layer depth from the point of view of the surface influence on temperature profile variations) is Λ = 100 m - 3 km and typically 500 - 1000 m. It is in correspondence with conventional values for the atmosphere boundary layer depth.

V. Conclusion.

New results in the theory of simultaneous solution of emission transfer and thermal conductivity equations have been obtained, in particular, the relation between brightness temperatures at two different wavelengths and expression for heat flux in half-space as functional of its brightness temperature dynamics.

On the basis of results of simultaneous solution of emission transfer and thermal conductivity equations the stochastic theory of temperature distribution and thermal radioemission of the medium (half-space) has been developed. The equations for covariance and autocovariance functions for temperature profile and brightness temperatures of thermal radioemission have been obtained. These expressions have been applied for estimations of correspondent atmosphere and soils parameters.

The developed theory gives wide possibilities for radiometer investigation of atmosphere-underground system. New results can be obtained on the base of numerical calculation of statistical parameters and its comparison with the same parameters obtained from measurements data and from the results of temperature retrieval on the base of Tikhonov's method (see, for example, [1,3]). These investigations will establish the application possibilities and restrictions of this theory.

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