Stochastic approach to results of simultaneous solution of emission transfer and thermal conductivity equations [1]

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Absrtact--On the basis of results of simultaneous solution of thermal emission transfer and thermal conductivity equations the stochastic theory of temperature distribution and thermal radioemission of medium (half-space) has been developed. Expressions for covariance functions of temperature profile and brightness temperature as functions of statistical parameters of half-space surface temperature which was considered as a random function of time have been obtained.

INTRODUCTION

In previous works [1-2] the theory of radioemission in medium (half-space) with temperature distribution which depends on boundary conditions (surface temperature or heat flux) dynamics has been developed. On the base of simultaneous solution of emission transfer and thermal conductivity equations it appeared possible to obtain expressions for brightness temperature of radioemission as integrals of boundary conditions evolution [1], and next, to make inversion of these expressions and to obtain formulas for boundary conditions and temperature distribution (profile) of the medium as integrals of brightness temperature evolution [2]. These results have been applied for radiometer investigations of diurnal temperature and heat dynamics in soil (by brightness temperature measurements at wavelengths 0.8 and 3 cm), and also for investigations of atmosphere boundary layer (by brightness temperatures at wavelength 0.5 cm in the oxygen line center).

But the results of simultaneous solution of emission transfer and thermal conductivity equations can be used not only as a new retrieval method, but also to develop the stochastic theory of the medium, if one considers surface temperature as a random function of time.

PROBLEM FORMULATION

Let us consider the homogeneous half-space $z \leq 0$ with the constant parameters: thermal diffusivity coefficient a^2 and absorption (of thermal radioemission) coefficient γ. If we have boundary condition for temperature $T(0,t) = T_0(t)$, then the temperature distribution inside the half-space can be determined from thermal conductivity equation as a function of depth and time. The brightness temperature $T_B(\lambda)$ of upward thermal radioemission at wavelength λ is determined from the known solution of emission transfer equation.

The simultaneous solution of these equations gives the expression for brightness temperature as functional of surface temperature [1,2]:

$$T_B(t)=\int_{-\infty}^{t} T_0(\tau)\left[\frac{\gamma a}{\sqrt{\pi(t-\tau)}}-(\gamma a)^2 \operatorname{erfc}(\gamma a\sqrt{t-\tau})\, e^{(\gamma a)^2(t-\tau)}\right]d\tau . \quad (1)$$

The solution of (1) as Volterra's equation of the 1-st kind with the variable upper limit obtained in [2] can be expressed as

$$T_0(t) = T_B(t) + \frac{1}{\gamma a}\int_{-\infty}^{t} T_B'(\tau)\frac{d\tau}{\sqrt{\pi(t-\tau)}} =$$

$$=T_B(t) + \frac{1}{\gamma a}\int_{-\infty}^{t} (T_B(t) - T_B(\tau))\frac{d\tau}{\sqrt{\pi(t-\tau)^3}} . \quad (2)$$

This equation gives the solution of the problem of one-wavelength radiothermometry for homogeneous half-space:

$$T(z,t)=\int_{-\infty}^{t} T_B(\tau)(-z)\, e^{-\frac{z^2}{4a^2(t-\tau)}} \frac{d\tau}{\sqrt{4\pi a^2(t-\tau)^3}} +$$

$$\frac{1}{\gamma a}\int_{-\infty}^{t} T_B'(\tau)\, e^{-\frac{z^2}{4a^2(t-\tau)}} \frac{d\tau}{\sqrt{\pi(t-\tau)}} . \quad (3)$$

Performing the integration of the second term in (3) by parts, one has [2]

$$T(z,t)=\int_{-\infty}^{t} T_B(\tau)\, e^{-\frac{z^2}{4a^2(t-\tau)}}\left[\frac{1}{\gamma}\left(\frac{z^2}{2a^2(t-\tau)}-1\right)-z\right]\frac{d\tau}{\sqrt{4\pi a^2(t-\tau)^3}} . \quad (4)$$

The above expression is valid for all values of z with the exception of $z = 0$ where it is impossible to perform the

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integration by parts in (3). Next, it is easy to obtain the formula which expresses the brightness temperature at one wavelength as a functional of evolution of brightness temperature at another wavelength:

$$T_{B2}(t)=\frac{\gamma_2}{\gamma_1}T_{B1}(t)+\int_{-\infty}^{t}T_{B1}(\tau)\cdot \left(1-\frac{\gamma_2}{\gamma_1}\right)\left(\gamma_2 a\left[\frac{1}{\sqrt{\pi(t-\tau)}}-\gamma_2 a e^{(\gamma_2 a)^2(t-\tau)}\,erfc\left(\gamma_2 a\sqrt{t-\tau}\right)\right]\right)d\tau \tag{5}$$

where γ_1, γ_2 - absorption coefficients at wavelengths λ_1 and λ_2 respectively.

STOCHASTIC THEORY OF HALF-SPACE

If the surface temperature is a random function, then, using the fact that all these integral expressions are linear, it is possible to develop the stochastic theory for the random components of temperature distribution and thermal radioemission of the medium on the basis of known approach in the theory of stationary random processes for linear systems which leads to Wiener-Li expressions. The property of covariance functions
$B_{yx}(-\tau)=B_{xy}(\tau)$ is also in use.

Now, let us consider the boundary condition for the temperature as a random stationary function with the middle value $\langle T_0\rangle$, mean square deviation σ_{T_0} and autocovariance function
$B_{T_0T_0}(\tau)=\langle (T_0(t)-\langle T_0\rangle)(T_0(t+\tau)-\langle T_0\rangle)\rangle$ which for simple evaluations will be used in the form:

$$B_{T_0T_0}(\tau)=\sigma_{T_0}^2\exp\left(-\left|\frac{\tau}{\tau_0}\right|\right) \tag{6}$$

where τ_0 is the correlation time.

It is clear that for mean values
$\langle T(z)\rangle = \langle T_0\rangle$, $\langle T_B\rangle = \langle T_0\rangle$ because of the unity normalization of correspondent integral expressions. So, from solution of thermal conductivity equation we have expressions for covariance function between surface temperature and temperature $T(z)$ at level z

$$B_{T_0T}(\tau)={}_0\int^{\infty}B_{T_0T_0}(\tau-\tau')\frac{|z|}{2\sqrt{\pi}a}e^{-\frac{z^2}{4a^2\tau'}}\frac{d\tau'}{(\tau')^{3/2}}, \tag{7}$$

for mean square variance

$$\sigma_T^2(z)={}_0\int^{\infty}{}_0\int^{\infty}B_{T_0T_0}(\tau-\tau')\frac{z^2}{4\pi a^2}e^{-\frac{z^2}{4a^2}\left(\frac{1}{\tau'}+\frac{1}{\tau}\right)}\frac{d\tau d\tau'}{(\tau\tau')^{3/2}}, \tag{8}$$

for autocovariance function of temperature at level z

$$B_{TT}(\tau)={}_0\int^{\infty}{}_0\int^{\infty}B_{T_0T_0}(\tau'-\tau''-\tau)\frac{z^2}{4\pi a^2}\cdot\frac{z^2}{4\pi a^2}e^{-\frac{z^2}{4a^2}\left(\frac{1}{\tau'}+\frac{1}{\tau''}\right)}\frac{d\tau' d\tau''}{(\tau'\tau'')^{3/2}}, \tag{9}$$

and also for interlevel covariance function between temperature variances T_1 at the level z_1 and T_2 at the level z_2 :

$$B_{T_2T_1}(z_2,z_1,\tau)={}_0\int^{\infty}{}_0\int^{\infty}{}_0\int^{\infty}B_{T_0T_0}(\tau'-\tau''+\tau'''-\tau)\frac{z_2^2|z_1-z_2|}{8\pi^{3/2}a^3}\cdot e^{-\frac{z_2^2}{4a^2}\left(\frac{1}{\tau'}+\frac{1}{\tau''}\right)-\frac{(z_1-z_2)^2}{4a^2}\frac{1}{\tau'''}}\frac{d\tau' d\tau'' d\tau'''}{(\tau'\tau''\tau''')^{3/2}}\cdot \tag{10}$$

The covariance function between surface temperature of the medium and brightness temperature of its radioemission can be expressed as

$$B_{T_0T_B}(\tau)={}_0\int^{\infty}B_{T_0T_0}(\tau-\tau')\frac{\gamma a}{\sqrt{\pi\tau'}}\cdot\frac{\gamma a}{\sqrt{\pi\tau'}}[1-\sqrt{\pi}(\gamma a)\sqrt{\tau'}e^{(\gamma a)^2\tau'}erfc(\gamma a\sqrt{\tau'})]d\tau', \tag{11}$$

for mean square variance

$$\sigma_{T_B}^2={}_0\int^{\infty}{}_0\int^{\infty}B_{T_0T_0}(\tau-\tau')\frac{(\gamma a)^2}{\pi\sqrt{\tau\tau'}}\cdot[1-\sqrt{\pi}(\gamma a)\sqrt{\tau'}e^{(\gamma a)^2\tau'}erfc(\gamma a\sqrt{\tau'})]\cdot[1-\sqrt{\pi}(\gamma a)\sqrt{\tau}e^{(\gamma a)^2\tau}erfc(\gamma a\sqrt{\tau})]d\tau d\tau', \tag{12}$$

and for brightness temperature autocovariance function

$$B_{T_BT_B}(\tau)={}_0\int^{\infty}{}_0\int^{\infty}B_{T_0T_0}(\tau'-\tau''-\tau)\frac{(\gamma a)^2}{\pi\sqrt{\tau'\tau''}}\cdot \tag{13}$$

$$\cdot[1-\sqrt{\pi}(\gamma a)\sqrt{\tau'}e^{(\gamma a)^2\tau'}erfc(\gamma a\sqrt{\tau'})][1-\sqrt{\pi}(\gamma a)\sqrt{\tau''}e^{(\gamma a)^2\tau''}erfc(\gamma a\sqrt{\tau''})]d\tau' d\tau''$$

It is also possible to obtain the expression for covariance function between brightness temperatures T_{B1} and T_{B2} at two different wavelengths λ_1 and λ_2 :

$$B_{T_{B_1}T_{B_2}}(\lambda_1,\lambda_2,\tau)=\frac{\gamma_2}{\gamma_1}B_{T_{B_1}T_{B_1}}(\tau)+(1-\frac{\gamma_2}{\gamma_1})\int_0^\infty B_{T_{B_1}T_{B_1}}(\tau-\tau')\cdot \cdot\frac{\gamma_1 a}{\sqrt{\pi\tau'}}[1-\sqrt{\pi}(\gamma_1 a)\sqrt{\tau'}e^{(\gamma_1 a)^2\tau'}erfc(\gamma_1 a\sqrt{\tau'})]d\tau' \quad (14)$$

and to obtain the formula for for covariance function between brightness temperature and temperature at the level z :

$$B_{T_BT}(\tau)=\int_0^\infty B_{T_BT_B}(\tau-\tau')\frac{|z|}{2\sqrt{\pi}a}e^{-\frac{z^2}{4a^2\tau'}}\frac{d\tau'}{(\tau')^{3/2}}+ +\frac{1}{\gamma a}\int_0^\infty\frac{\partial B_{T_BT_B}}{\partial\tau'}(\tau-\tau')e^{-\frac{z^2}{4a^2\tau'}}\frac{d\tau'}{\sqrt{\pi}(\tau')^{1/2}}. \quad (15)$$

From expressions for correlation functions one can see that these functions are not symmetric relative to the point $\tau=0$ and, moreover, don't achieve its maxima at this point, i.e. the prediction for the future is not symmetric relative to the prediction for the the past.

For exponential covariance function (6) some simple results can be obtained. The expression (7) yields

$$B_{T_0T}(\tau)=\sigma_{T_0}^2 e^{-\frac{|z|}{a\sqrt{\tau_0}}-\left|\frac{\tau}{\tau_0}\right|}, \qquad \tau\le 0, \quad (16)$$

$$=\frac{\sigma_{T_0}^2|z|}{2\sqrt{\pi}a}[\int_0^\tau e^{-\frac{z^2}{4a^2\tau}+\frac{\tau'}{\tau_0}}\frac{d\tau'}{(\tau')^{3/2}}\cdot e^{-\frac{\tau}{\tau_0}}+ +\int_\tau^\infty e^{-\frac{z^2}{4a^2\tau}-\frac{\tau'}{\tau_0}}\frac{d\tau'}{(\tau')^{3/2}}\cdot e^{\frac{\tau}{\tau_0}}],\tau>0.$$

In particular,

$$B_{T_0T}(0)=\sigma_{T_0}^2 e^{-\frac{|z|}{a\sqrt{\tau_0}}}. \quad (17)$$

One can see that there is a correlation depth $\Lambda = a\sqrt{\tau_0}$ which can be considered as one of definitions for boundary layer. From (11) it is possible to obtain

$$B_{T_0T_B}(\tau)=\sigma_{T_0}^2\frac{\sqrt{\frac{\tau_0}{\Gamma}}}{1+\sqrt{\frac{\tau_0}{\Gamma}}}e^{-\left|\frac{\tau}{\tau_0}\right|}, \ \tau\le 0, \quad (18)$$

$$=\sigma_{T_0}^2\{e^{-\frac{\tau}{\tau_0}}+\frac{1}{\tau_0}e^{-\frac{\tau}{\tau_0}}\frac{1}{[(\gamma a)^2+\frac{1}{\tau_0}]}[\gamma a\sqrt{\tau}{}_1F_1(\frac{1}{2},\frac{3}{2},\frac{\tau}{\tau_0})+e^{-[(\gamma a)^2+\frac{1}{\tau_0}]\tau}erfc(\gamma a\sqrt{\tau})-1]-$$

$$-\frac{2}{\tau_0}e^{\frac{\tau}{\tau_0}}\frac{1}{[(\gamma a)^2-\frac{1}{\tau_0}]}[\gamma a\sqrt{\tau_0}erfc(\frac{\tau}{\tau_0})-e^{[(\gamma a)^2-\frac{1}{\tau_0}]\tau}erfc(\gamma a\sqrt{\tau})]\}$$

$\tau > 0$,

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where ${}_1F_1$ is the singular hypergeometric function, $\Gamma = 1/(\gamma a)^2$ is time parameter which determines the time of the heating at the skin depth $d=1/\gamma$. The expression (18) yields:

$$B_{T_0T_B}(0)=\sigma_{T_0}^2\frac{\sqrt{\frac{\tau_0}{\Gamma}}}{1+\sqrt{\frac{\tau_0}{\Gamma}}}. \quad (19)$$

CONCLUSION

Stochastic theory of half-spase temperature and thermal radio emission based on results of simultaneous solution of emission transfer and thermal conductivity equations has been devalopeded. For the case of exponential covariance function of boundary temperature simple and physically clear formulas have been obtained.

REFERENCIES

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