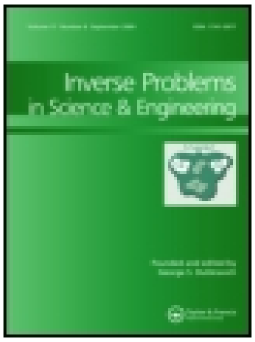

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Dual regularization in non-linear inverse scattering problems

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ABSTRACT
A new method in the theory of non-linear ill-posed problems is adapted and applied to various one-dimensional inverse scattering problems of electromagnetic subsurface diagnostics of permittivity inhomogeneities. Based on the developed theory, solving algorithms have been worked out and applied in problems of low-frequency diagnostics of conductivity profile in geomagnetic exploration, microwave monitoring of water diffusion in soil and reflectometry diagnostics of inhomogeneities in multilayer structures of X-ray optics – covering the scale range from nanometers to kilometres. Results demonstrate new possibilities of developed approach in these applications.

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1. Introduction

Inverse problems of scattering are widely used in various methods of sounding and tomography of media parameters in electromagnetism, acoustics and quantum mechanics. A.N. Tikhonov was the first who proposed to use low-frequencies electromagnetic measurements in the geomagnetic exploration,[1] followed by L. Cagniard [2] and J. R. Wait [3]. Then, various methods to solve this inverse problem (modelling, parameterization, statistical regularization, gradient minimization of discrepancy and reduction to integral equation – see, for example, in [4–9]) have been proposed.

However, notwithstanding to multiple approaches that has been developed in various applications, the non-linear problems of the subsurface electromagnetic diagnostics considered here have no universal rigorous solution by now (some one-dimensional problems can be reduced to Gelfand–Levitan–Marchenko equation that is solved explicitly,[10] but this theory is inapplicable to layered or absorbing media). Algorithms that have been used in practice are based mostly on parameterization, but they have no convergence and lead to wrong results in cases when the chosen parameterization appears unsuitable for the real inhomogeneity. It is well known that such common-sense (according Tikhonov's definition) regularization methods are widely in use in applications. They are based on approximations of the desired solution by simple functions with

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few unknown parameters, or use its decompositions (Fourier, polynomial and other series, eigenfunctions, etc.) that lead to a lower dimension of the desired solution and corresponding system of equations to be solved. Restrictions of such 'regularizations' are quite obvious; proper explanations, examples and demonstration can be found, for example, in [9].

Tikhonov's method of generalized discrepancy is effective and has a strong convergence in linear problems.[11] In the solution of some non-linear problems, this method also has been applied successfully, for example, in retrieval of the atmosphere ozone profile by multifrequency radiometry data.[9] Also, some approaches based on this method have been worked out for inverse scattering problems formulated for non-linear integral equations that have been solved iteratively, as a sequence of linear Fredholm integral equations of the first kind, beginning with the Born approximation.[12–15] Good results have been demonstrated in application of this approach to the subsurface tomography of low-contrast inhomogeneities.[15] However, in this problem as well as in application to problems that are considered in this paper, corresponding algorithms demonstrated good results only in cases of low-contrast inhomogeneities or if there is a good first guess. Results of the numerical study revealed serious limitations for large perturbations, when the Born approximation (first guess of iterative methods) gives poor results.[16]

Because of these restrictions, there is a need in more powerful methods of modern theory of non-linear ill-posed problems. Notwithstanding the absence of some universal methods in this theory, there are approaches that can be successfully applied in practice (see in books [17–19] and concrete methods in [20–25]). We apply here a new method of dual regularization based on the Lagrange approach in the general optimization theory [23–25] that has been firstly proposed as a possible approach to inverse scattering problems in [16]. In this paper, it is applied to the solution of three one-dimensional inverse scattering problems taking into account their specific character: (*a*) to the problem of the low-frequency sounding of the Earth crust; (*b*) to the problem of the microwave monitoring of water content profiles in the process of water diffusion in soil; (*c*) to the problem of X-ray diagnostics of permittivity inhomogeneities in multilayer periodical structures of modern X-ray optics. The developed algorithms are studied in numerical simulation, and, in the problem (*b*), they are also applied to experimental data. We consider this study as the first step to the solution of more difficult three-dimensional problems of subsurface microwave tomography proposed in [13–15].

2. Method of dual regularization

In most of inverse problems of physical diagnostics,[9] it is convenient to transform the initial problem formulated in terms of differential equations to the statement based on the solution of integral equations, typically integral equations of the first kind (Fredholm or Volterra). There are effective methods, such as Tikhonov's methods of generalized discrepancy and of regularization on compact sets [11] to solve these problems. Whereas, the reduction to integral equations is a proper way in linear problems, in non-linear problems, the transfer from the differential statement to its 'integral analogue' can be a new independent and, mostly, more complicated problem. To make such a transition, one has to use various approximations (for, example, Born approximation). At that, errors appear, and the initial physical model may be essentially distorted.

However, this transfer is not indispensable, and it can be eliminated from the procedure of solving using available methods in the theory of non-linear ill-posed problems. Here, we use the method of dual regularization [22–24] based on the theory of optimization and optimal control. It is stable to data errors and suitable for the solution of non-linear problems. It develops a similar approach that has been worked out early for the solution of convex linear problems [25] based on the classical idea of removing constraints, which underlies the Lagrange principle. Two problems are solved simultaneously in dual algorithms: the initial problem and the problem which is dual to it. At that, solving the dual problem (that is always the problem of convex optimization) leads to constructive approximation of the solution of the initial problem. Solving the initial optimization problem in aggregate with solving the dual problem forms a saddle point of the Lagrange function of the initial problem.

Probably, the first dual algorithm in the optimization theory has been proposed by Uzawa in 1958 [26] (see also in [27]). It was based on the gradient solution of the dual problem, and has gained a wide popularity. However, the corresponding convergence theorems [28–30] contain two important assumptions: (i) data are free from errors; (ii) the Lagrange function has the saddle point. Both assumptions are rather restrictive, because data errors are inevitable in real applications and the proof of the existence or non-existence of a saddle point in such problems is generally a difficult mathematical problem. Generally, the formal application of the algorithm may lead and does lead to the instability of the approximate solution.[25]

In 1968, works of M. Hestens and M. Powell [31,32] give a stimulus to the further development of dual methods for non-linear problems with restrictions. Details of dual methods in non-linear optimization problems of mathematical programming in finite-dimensional spaces (called the Lagrange multiplier methods) can be found in [27,33–35] and included bibliography. The basic idea of these methods is directly connected with the use of so-called modified Lagrange function that is the sum of classical Lagrange function and penalty term with a positive penalty coefficient that can be formed by a variety of ways. As it is typical for non-linear problems, the main feature of dual methods based on the modified Lagrange problems is a local convergence, i.e. they converge, if the first approximation for the pair of both variables (direct and dual) is close enough to the desired optimal pair. The necessary conditions of this convergence are formulated as some a priori suppositions about input data in terms of the desired optimal point x^*, and, because of this reason, are unverifiable.

The considered inverse problems can be expressed formally as the problem of solving the non-linear operator equation

$$g^{\delta}(z) = 0, \quad z \in D \subset Z, \tag{1}$$

where g^{δ}: $D \to H$ is a weekly continuous operator with the range of values $g(D)$ that is a compact in H, $D \subset Z$ is a closed-convex bounded set, Z, H are the Hilbert spaces. Here and below, the upper index $\delta > 0$ marks the deviation of perturbed input data from the exact values $g^0 (\delta = 0)$. For the problem (1), the estimation of the deviation satisfies to inequality

$$\|g^{\delta}(z) - g^{0}(z)\| \leq C\delta \quad \forall z \in D.$$

The reduction of the ill-posed problem (1) to the optimization problem is one of the general approaches. Following this approach, instead of the problem (1), let us consider the problem of minimization

$$\|z\|^2 \to \min, g^\delta(z) = 0, \quad z \in D \subset Z, \tag{2}$$

i.e. as it is conventional in the ill-posed theory, let us consider the ill-posed problem to find a minimum-norm solution z^0 of (1) for $\delta = 0$. Such a solution exists for certain, if the set of solutions of (1) isn't empty; however, the uniqueness of this solution isn't guaranteed.

In solving the problem (2) that is unstable to errors of input data, we apply here the method of dual regularization.[22,23,25,36,37] According to this method, a maximizing sequence that includes the Tikhonov's regularization can be constructed for the dual variable in the problem dual to (2). Simultaneously with this process, the minimizing approximate solution of the problem (2) in the sense of J. Warga [39] is constructed. Initially, it has been proposed to the problem of convex minimization

$$f^\delta(z) \to \min, \quad g^\delta(z) \equiv A^\delta z - h^\delta = 0, \quad z \in D \subset Z \tag{3}$$

with the continuous strongly convex functional $f^\delta: D \to R^1$, convex closed set D, linear bounded operator $A^\delta: Z \to H$ and given element $h^\delta \in H$. Along with the problem (3), it is possible to present its parametric version with the parameter $p \in H$in the constraint

$$f^\delta(z) \to \min, \quad g^\delta(z) \equiv A^\delta z - h^\delta = p, \ z \in D \subset Z$$

and with the lower semi-continuous value function $\beta: H \to R^1 \cup \{+\infty\}$

$$\beta(p) \equiv \lim_{\varepsilon \to 0} \equiv \beta_\varepsilon(p), \ \beta_\varepsilon(p) \equiv \inf\{f^0(z): z \in D_p^{o,\varepsilon}\}, \ \beta_\varepsilon(p) \equiv +\infty \text{ at } D_p^{o,\varepsilon} = \emptyset,$$

where $D_p^{o,\varepsilon} \equiv \{z \in D: \|g^0(z) - p\| \le \varepsilon\}, \ \varepsilon \ge 0.$

From the formal viewpoint, the dual regularization method [25,36,37] as applied to the convex problem (3) consists in the direct solving the exact ($\delta = 0$) problem, dual to (3), with Tikhonov's regularization:

$$V^\delta(\lambda) \to \sup, \quad \lambda \in H, \quad V^\delta(\lambda) \equiv \inf_{z \in D} L^\delta(z, \lambda), \tag{4}$$

$$L^\delta(z, \lambda) \equiv f^\delta(z) + \left\langle \lambda, A^\delta z - h^\delta \right\rangle$$

Approximation of the unique solution $z^0 \in D$ of the initial exact problem (the construction of the minimizing approximate solution in the sense of J. Warga [39]) is realized (see in details in [20,31,32]) at the condition of the coordinate approach of the regularization parameter α and the parameter of input data errors δ

$$\left|f^\delta(z) - f^0(z)\right| \le C\delta \ \forall z \in D, \ \ \left\|A^\delta z - A^0 z\right\| \le C\delta(1 + \|z\|) \ \forall z \in Z, \ \left\|h^\delta - h^0\right\| \le C\delta.$$

to zero in the regularized dual problem

$$V^\delta(\lambda) - \alpha\|\lambda\|^2 \to \sup, \quad \lambda \in H. \tag{5}$$

As a result of this procedure that is formally the regularization process of the maximizing sequence construction in the exact problem (4), the exact solution z^0 of (3) is approximated by elements $z^\delta[\lambda^{\delta,\alpha(\delta)}] \equiv \operatorname{argmin}\{L^\delta(z, \lambda^{\delta,\alpha(\delta)}) : z \in D\}$, where $\lambda^{\delta,\alpha(\delta)} \equiv \operatorname{argmax}\{V^\delta(\lambda) - \alpha(\delta)\|\lambda\|^2 : \lambda \in H\}$.

In the case of subdifferentiability of the convex lower semi-continuous function β in the point $p = 0$ that is equivalent to the solvability of the unperturbed ($\delta = 0$) dual problem (4); elements $\lambda^{\delta,\alpha(\delta)}$ that are solutions of the regularized dual problem (5) are strongly convergent to the minimum-norm solution of (4). If $\partial\beta(0) = \varnothing$, then $\left\|\lambda^{\delta,\alpha(\delta)}\right\| \to +\infty$ at $\delta \to 0$. Simultaneously, elements $z^\delta[\lambda^{\delta,\alpha(\delta)}]$ are strongly converging to the unique solution $z^0$of unperturbed problem (3) independently of whether the subdifferential $\partial\beta(0)$ is vacuous or not. Thus, in the convex case, the convergence properties of the dual regularization method are determined by the subdifferential properties of the function β in the point $p = 0$ in the sense of convex analysis. Simultaneously, this subdifferentiability is indissolubly connected with the classical construction of the Lagrange function $L^\delta(z, \lambda)$ (see in detail in [20,31,32]).

In the case of the non-linear problem (3), convergence properties of the non-linear dual regularization method [17–19,33] are also completely determined by the subdifferentiability of the value function β in the parametric problem

$$\|z\|^2 \to \min, g^\delta(z) = p, \quad z \in D \subset Z$$

that is determined similar to the convex case [25,36,37] as

$$\beta(p) \equiv \lim_{\varepsilon \to +0} \beta_\varepsilon(p), \ \beta_\varepsilon(p) \equiv \inf\{\|z\|^2 : z \in D_p^{o,\varepsilon}\}, \ \ D_p^{o,\varepsilon} = \{z \in D : \left\|g^0(z) - p\right\| \le \varepsilon\}.$$

However, the subdifferentiability of this lower semi-continuous but, in general, non-convex function $\beta : H \to R^1 \cup \{+\infty\}$, should be understood in the sense of non-convex (non-smooth) analysis.[40–42] In this analysis, in the capacity of the concept of non-convex sub differentiability,[22–24,38] concepts of proximal subgradient [40,41] and of subdifferential Frechet [45,47] of lower semi-continuous functions in the Hilbert space are in use. Before outlining the background related to this concept, we recall the definition of the proximal normal.

Definition 1. (a) Let H be a Hilbert space, $S \in H$ be a closed set and $\bar{s} \in S$. A vector $\varsigma \in H$ is said to be the *proximal normal* to S at the point $\bar{s} \in S$, if there exists a constant $M > 0$ such that

$$\langle \varsigma, s - \bar{s} \rangle \le M\|s - \bar{s}\|^2 \quad \forall s \in S. \tag{6}$$

The set of all such vectors ς, which represents a cone, is denoted by $\hat{N}_S(\bar{s})$ and is called a proximal normal cone.

(b) Let $f : H \to R^1 \cup \{+\infty\}$ be a lower semi-continuous function and $\bar{x} \in \operatorname{dom} f$. A vector $\varsigma \in H$ is said to be the *proximal subgradient* of the function f at the point $\bar{x}$ if $(\varsigma, -1) \in \hat{N}_{\operatorname{epi} f}(\bar{x}, f(\bar{x}))$. The set of all such vectors ς is denoted by $\partial_p f(\bar{x})$ and is referred as the proximal subgradient of the function f in the point $\bar{x}$.

Below the necessary and sufficient condition [40,41] for a vector to be the proximal subgradient of a lower semi-continuous function at a given point is formulated.

Lemma 1. Let H be a Hilbert space, $f : H \to R^1 \cup \{+\infty\}$ be a lower semi-continuous function, and $\bar{x} \in \operatorname{dom} f$. A vector $\varsigma \in H$ is the proximal subgradient of the function f at $\bar{x}$, i.e. $\varsigma \in \partial_p f(\bar{x})$, if and only if there exist constants $R > 0$ and $\delta > 0$ such that

$$f(\bar{x}) - \langle \varsigma, \bar{x} \rangle \le f(x) - \langle \varsigma, x \rangle + R\|x - \bar{x}\|^2 \ \forall x \in S_\delta(\bar{x}) \equiv \{y \in H : \|y - \bar{x}\| < \delta\}. \tag{7}$$

Let us recall the concept of the Frechet normal to a closed set in a Banach space and define the corresponding Frechet subdifferential of a lower semi-continuous function (see [40,42]).

Definition 2. Let Ω be a non-empty subset of a Banach space X. Let $\bar{x} \in \mathrm{cl}\,\Omega$ and $u \xrightarrow{\Omega} \bar{x}$ means that $u \to \bar{x}$ with $u \in \Omega$. Then the non-empty cone set

$$\hat{N}(x;\Omega) \equiv \left\{ x^* \in X^*\colon \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le 0 \right\} \tag{8}$$

is called the Frechet normal cone to Ω at the point x and is denoted by $\hat{N}(x;\Omega)$.

Definition 3. Let $f\colon X \to R^1 \cup \{+\infty\}$ be a lower semi-continuous function defined on a Banach space X, $\bar{x} \in \mathrm{dom}\, f$. The set

$$\hat{\partial} f(\bar{x}) \equiv \{x^* \in X^*\colon (x^*, -1) \in \hat{N}((\bar{x}, f(\bar{x})); \mathrm{epi}\, f)\} \tag{9}$$

is called the Frechet subdifferential of f at the point $\bar{x}$. If $\bar{x} \notin \mathrm{dom}\, f$, we set $\partial_p f(\bar{x}) = \emptyset$.

Lemma 2. Let $f\colon H \to R^1 \cup \{+\infty\}$ be a lower semi-continuous function defined on a Banach space X, $\bar{x} \in \mathrm{dom}\, f$. Then $x^* \in \hat{\partial} f(x)$ if and only if for every $\varepsilon > 0$ there exists a neighbourhood X_ε such that

$$f(x) - \langle x^*, x\rangle \le f(x') - \left\langle x^*, x' \right\rangle + \varepsilon \left\| x' - x \right\| \; \forall x' \in X_\varepsilon. \tag{10}$$

An important property of lower semi-continuous functions $f\colon H \to R^1 \cup \{+\infty\}$ is that both the set $\partial_p f(x)$ (in the case of Hilbert space X) and the set $\hat{\partial} f(x)$ (in the space X from a rather vide class of Banach spaces) (for more details see, e.g. [41,42]) are both not empty for a dense set of points in $\mathrm{dom}\, f$. In this paper, we assume that X is a Hilbert space for which the above-mentioned properties are fulfilled.

In [23,24,38], where non-linear problems more general as compared to (2) are considered, it was shown that the non-emptiness of the proximal subgradient $\partial_p \beta(0)$ generates the corresponding construction of the modified Lagrange function in the problem (2)

$$L_c^{\delta,2}(z, \lambda) \equiv \|z\|^2 + \left\langle \lambda, g^\delta(z) \right\rangle + c\left\| g^\delta(z) \right\|^2, \quad z \in D, \lambda \in H, \tag{11}$$

where c is a penalty coefficient large enough. In its turn, the non-emptiness of the Frechet subdifferential $\hat{\partial}\beta(0)$ inevitably leads to the modified Lagrange function in the problem (2)

$$L_c^{\delta,1}(z, \lambda) \equiv \|z\|^2 + \left\langle \lambda, g^\delta(z) \right\rangle + c\left\| g^\delta(z) \right\|, \quad z \in D, \lambda \in H, \tag{12}$$

where c is also a penalty coefficient large enough. So, it is convenient to combine these two cases in a common case, and to determine the mixed construction of the modified Lagrange function

$$L_c^{\delta}(z, \lambda) \equiv \|z\|^2 + \left\langle \lambda, g^\delta(z) \right\rangle + c(l_1\left\| g^\delta(z) \right\| + l_2\left\| g^\delta(z) \right\|^2), \quad z \in D, \lambda \in H, \tag{13}$$

where $l_1, l_2 \in \{0, 1\}$ are the weight coefficients. It is this construction of the modified Lagrange function that is in use in this paper.

In [23,24,38], the corresponding concept of the generalized Kuhn–Tucker vector of the problem (2) is introduced. It is a vector $\lambda \in H$, such that $\beta(0) \le L^\delta_c(z,\lambda)\ \forall z \in D$ In [23,24,38] the modified dual problem $V^\delta_c(\lambda) \to \sup,\quad \lambda \in H,\quad V^\delta_c(\lambda) \equiv \inf_{z\in D} L^\delta_c(z,\lambda)$ is also introduced. The approximation of the solution $z^0 \in D$ (non-unique, in general) of initial non-linear exact problem (2), or, in other words, the construction of minimizing approximate solution in the sense of J. Warga, is realized, as in the convex problem, at the condition of a consistent approach of the regularization parameter α to zero in the regularized dual problem

$$V^\delta_c(\lambda) - \alpha\|\lambda\|^2 \to \sup, \quad \lambda \in H \tag{14}$$

and the value of error parameter of input data δ in the problem (2). In the problem (2), it is possible to obtain the explicit form for superdifferential

$$\partial(V^\delta_c(\lambda) - \alpha\|\lambda\|^2) = \text{cl conv}\{g^\delta(z^*): z^* \in \text{Argmin}\{L^\delta_c(z,\lambda): z \in D\}\} - 2\alpha\lambda, \tag{15}$$

that is necessary to build the regularizing process of maximization in the dual problem. Here, the set $\text{Argmin}\{L^\delta_c(z,\lambda): z \in D\}$ is not empty for $\lambda \in H$ due to conditions concerning input data in the problem (2).

In the assumption of the weak continuity of the operator g^δ (avoiding details that can be found in [22–24,38]), it is possible to prove the theorem about the convergence of the non-linear version of the dualregularization method:

Theorem 1. Let $\delta^s,\ s = 1,2,\ldots$ be an arbitrary sequence of positive numbers converging to zero, and $\alpha(\delta^s) \to 0,\ \delta^s/\alpha(\delta^s) \to 0, s \to \infty$. If the problem (2) has a Kuhn–Tucker vector in the above-noted generalized sense, then, in the supposition that at least one of two penalty coefficients l_1, l_2 is positive, there is a number $c > 0$ large enough so that the next limiting relations are fulfilled:

$$\|z^s\|^2 \to \beta(0) = \|z^0\|^2,\ g^0(z^s) \to 0,\ \lambda^{\delta^s,\alpha(\delta^s)} \to \lambda^0_c,\ V^0_c(\lambda^{\delta^s,\alpha(\delta^s)}) \to \beta(0),\ s \to \infty, \tag{16}$$

where z^s, $s = 1, 2, \ldots$ are elements minimizing (at $\kappa > 0$) the modified Lagrange function $L^{\delta^s}_{c+\kappa}(z, \lambda^{\delta^s,\alpha(\delta^s)}),\ z \in D, \lambda^{\delta^s,\alpha(\delta^s)}, s = 1,2,\ldots$ are elements maximizing the strongly convex functional $V^{\delta^s}_c(\lambda) - \alpha(\delta^s)\|\lambda\|^2, \lambda \in H$ on the set $\Lambda_c \equiv \{\lambda \in H: \|\lambda\| \le c\}$, λ^0_c is a minimum-norm generalized Kuhn–Tucker vector in the set Λ_c of the problem (2).

In the case, when this vector doesn't exist, in the supposition that coefficients l_1, l_2 are positive, for an arbitrary sequence converging to $+\infty$ of numbers c_s, $s = 1, 2, \ldots$ such that $c_s\delta^s \to 0, s \to \infty$, the next relations are fulfilled

$$\|z^s\|^2 \to \beta(0),\ g^0(z^s) \to 0,\ \left\|\lambda^{\delta^s,\alpha(\delta^s)}\right\| \to +\infty,\ V^0_c(\lambda^{\delta^s,\alpha(\delta^s)}) \to \beta(0),\ s \to \infty, \tag{17}$$

where z^s, $s = 1, 2, \ldots$ are elements minimizing the modified Lagrange function $L^{\delta^s}_{c^s}(z, \lambda^{\delta^s,\alpha(\delta^s)}),\ z \in D, \lambda^{\delta^s,\alpha(\delta^s)}, s = 1,2,\ldots$ are elements maximizing on the set $\Lambda_{c^s} \equiv \{\lambda \in H: \|\lambda\| \le c^s\}$ the strongly concave functional $V^{\delta^s}_{c^s}(\lambda) - \alpha(\delta^s)\|\lambda\|^2, \lambda \in H$.

In both above-considered cases from the number convergence $\|z^s\|^2 \to \beta(0) = \|z^0\|^2,\ s \to \infty$ and the weak compactness of the set D and H-property of the Hilbert space, it follows that any weak accumulation point of the sequence z^s, $s = 1, 2, \ldots$ is its strong accumulation point.

The most important feature of this non-linear algorithm in the construction of the minimizing sequence is the application of the Tikhonov's regularization [11] to solving the dual problem that provides the stability of the algorithm to data errors. This regularization in the process of maximization of the dual-concave function is a necessary procedure because of instability of infinite-dimensional problems, including concave problems. It should be stressed that only problems without data errors have been considered in previous papers related to the finite-dimensional Lagrange multiplier method cited above, and, hence, no regularization procedure has been proposed to solve ill-posed problems.

In problems of physical diagnostics considered in this paper, input data of the problem (2) include a finite error $\delta_R > 0$, so that $\left\| g^{\delta_R}(z^0) \right\| \le \delta_R$, where z^0 is the exact solution of the initial exact problem. At that, it is assumed that the sequence $\delta^k,\ k = 1, 2, \dots$ satisfies $\delta^1 > \delta_R$. The minimizing approximate solution of the problem (2) considered in Theorem 1 is constructed according to the stopping rule in the regularized iterative procedure of the gradient ascent in the regularized dual problem (14), using the explicit form for the superdifferential $\partial V_c^{\delta_R}(\boldsymbol{\lambda}^k)$ in (15) [22,23,38]:

$$\lambda^{k+1} = \lambda^k + \beta^k \partial V_c^{\delta_R}(\boldsymbol{\lambda}^k) - 2\beta^k \alpha^k \lambda^k, \quad k = 1, 2, \dots \tag{18}$$

To fulfil the convergence conditions of the approximate solution to the exact solution of the initial problem (2), sequences $\delta^k,\ \beta^k,\ \alpha^k$ should tend to zero in a coordinated way:

$$\delta^k \ge 0,\ \beta^k > 0,\ \alpha^k > 0, \lim_{k\to\infty}(\delta^k + \beta^k + \alpha^k) = 0, \frac{\alpha^k}{\alpha^{k+1}} < C_0, \tag{19}$$

$$\lim_{k\to\infty} \frac{\left|\alpha^{k+1} - \alpha^k\right|}{(\alpha^k)^2 \beta^k} = \lim_{k\to\infty} \frac{\beta^k}{\alpha^k} = \lim_{k\to\infty} \frac{\delta^k}{(\alpha^k)^3} = 0, \sum_{k=1}^{\infty} \alpha^k \beta^k = +\infty,$$

where $C_0 > 0$ is a constant.

To calculate the corresponding value of the superdifferential $\partial V_c^{\delta_R}(\lambda^k)$ at the every step of the iterative procedure (18), the problem of the minimization of the modified Lagrange function $L_c^{\delta_R}(z, \lambda^k) \to \min,\ z \in D$ in (13) is solved. This procedure continues up to the largest number $k = k(\delta_R)$, for which the inequity $\delta^k \ge \delta_R$ is fulfilled. At that, the corresponding solution of the minimization problem $L_c^{\delta_R}(z, \lambda^{k(\delta_R)}) \to \min,\ z \in D$ is assumed as the approximate solution of the problem (2).

Summing up, it is possible to make next assertions. In solving of non-linear inverse scattering problem, we apply the new method of dual regularization,[22,23] where the initial inverse problem is considered as a problem of non-linear programming in the infinite-dimensional Hilbert space with the infinite-dimensional equality-kind constrain. It makes possible: (*a*) to use well-known advantages of dual approaches for the solution of conditional minimization problems with linear constrains (see, for example, in [27]); (*b*) to use Tikhonov's stabilization, but only when solving the dual problem, that principally distinguishes the applied method from earlier developed dual methods in solving non-linear problems of finite-dimensional conditional minimization without regularization (see in [27,31–33] and included bibliography). As the closest to our work among those in the regularization theory we can mention papers.[43,44]

3. Fields in media and statement of inverse scattering problems

In media with the inhomogeneous distribution of complex permittivity $\varepsilon(\mathbf{r}) = \varepsilon_i + \varepsilon_1(\mathbf{r})$, in a quite generous case of an inhomogeneity $\varepsilon_1(\mathbf{r})$ in a multilayer medium with permittivity ε_i in ith layer, the complex amplitudes of vectors of electrical and magnetic fields $\mathbf{E}$, $\mathbf{H}$ [~exp $(-i\omega t)$] are determined by the complex amplitude of the source electric current density $\mathbf{j}$ from the Maxwell equations:

$$\nabla \times \mathbf{E} = \frac{i\omega}{c}\mathbf{H}, \tag{20}$$

$$\nabla \times \ \mathbf{H} + i\frac{\omega}{c}\varepsilon\mathbf{E} = \frac{4\pi}{c}\mathbf{j}, \tag{21}$$

where c the is light velocity, ω is the cyclic frequency. The contribution of $\varepsilon_1(\mathbf{r})$ in (20) can be considered as an effective source current $\mathbf{j}_{\text{eff}} = -\frac{i\omega}{4\pi}\varepsilon_1(z)\mathbf{E}$, so

$$\nabla \times \ \mathbf{H} + i\frac{\omega}{c}\varepsilon_i\mathbf{E} = -i\frac{\omega}{c}\varepsilon_1\mathbf{E} + \frac{4\pi}{c}\mathbf{j} = \frac{4\pi}{c}(\mathbf{j}_{\text{eff}} + \mathbf{j}). \tag{22}$$

Using the formalism of Green functions, the system of Equations (20) and (21) can be reduced to the non-linear integral equation [14]:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E_0}(\mathbf{r}) + \mathbf{E}_1(\mathbf{r}) = \mathbf{E_0}(\mathbf{r}) - \frac{i\omega}{4\pi}\int_V \overleftrightarrow{\mathbf{G}}(\mathbf{r}', \mathbf{r})\varepsilon_1(\mathbf{r}')\mathbf{E}(\mathbf{r}')d\mathbf{r}', \tag{23}$$

$$\mathbf{E}_0(\mathbf{r}) = \int_V \mathbf{j}(\mathbf{r}')\overleftrightarrow{\mathbf{G}}(x - x', y - y', z, z')d\mathbf{r}', \tag{24}$$

where the total electric field $\mathbf{E}(\mathbf{r})$ is expressed as the sum of the probing field $\mathbf{E_0}(\mathbf{r})$ and scattered field $\mathbf{E}_1(\mathbf{r})$. Equation (23) is a Fredholm integral equation of the second kind for solving the direct problem – calculation of $\mathbf{E}(\mathbf{r})$ that can be expressed by the Neumann series at the condition $\mathbf{E}_1(\mathbf{r}) << \mathbf{E_0}(\mathbf{r})$ in the range of inhomogeneities. The inverse scattering problem – to retrieve subsurface inhomogeneities $\varepsilon_1(\mathbf{r}')$ by measurements of scattered field – can be solved beginning with the Born approximation:

$$\mathbf{E}_1(\mathbf{r}) = -\frac{i\omega}{4\pi}\int_V \overleftrightarrow{\mathbf{G}}(\mathbf{r}', \mathbf{r})\varepsilon_1(\mathbf{r}')\mathbf{E}_0(\mathbf{r}')d\mathbf{r}'. \tag{25}$$

The convolution equation (24) for the probing field can be transformed in k-space over transversal co-ordinates:

$$\mathbf{E}_0(\kappa_x, \kappa_y, z) = 4\pi\int_{z'} \mathbf{j}(\kappa_x, \kappa_y, z')\overleftrightarrow{\mathbf{G}}(\kappa_x, \kappa_y, z, z')\mathrm{d}z', \tag{26}$$

where spectral components of Green tensors $\overleftrightarrow{\mathbf{G}}(\kappa_x, \kappa_y, z, z')$ have been obtained explicitly for arbitrary multilayer media in [14]. Various cases have been considered,[13–15] when

(25) can be reduced to convolution equations over transversal co-ordinates, and, then, using Fourier transform – to one-dimensional integral Fredholm equations of the first-kind relative to $\varepsilon_1(\kappa_x, \kappa_y, z)$ that should be solved for each pair κ_x, κ_y. Finally, the inverse Fourier transform of obtained k-space distributions gives the desired solutions of initial three-dimensional problems.

In cases of one-dimensional media, corresponding inverse problems are much simplified. Equation (23) in any such case can be reduced to the one-dimensional integral equation in k-space:

$$\mathbf{E}(\kappa_x, \kappa_y, z) = \mathbf{E}_0(\kappa_x, \kappa_y, z) - \frac{i\omega}{4\pi} \int\limits_{z'} \varepsilon_1(z') \overleftrightarrow{\mathbf{g}}(\kappa_x, \kappa_y, z', z) \mathbf{E}(\kappa_x, \kappa_y, z') \mathrm{d}z'. \quad (27)$$

However, the iterative solving of the inverse problem based on (13), as it was mentioned above, may lead to poor results for strong inhomogeneities. To solve such inverse problems in frameworks of the dual regularization method, it is reasonable to express the electric field on the interface of an inhomogeneous medium $z = 0$ from (27) as follows:

$$\mathbf{E}^{\parallel,\perp}(\kappa_x, \kappa_y, z = 0) = \mathbf{E}_0^{\parallel,\perp}(\kappa_x, \kappa_y, 0) + R^{\parallel,\perp}[\varepsilon_1](\kappa_x, \kappa_y) \mathbf{E}_0^{\parallel,\perp}(\kappa_x, \kappa_y, 0). \quad (28)$$

Then, the statement of such inverse problems can be based on the analysis of reflection coefficients at TH(∥) and TE(⊥) polarizations of plane waves or of some functional of these coefficients that can be calculated from initial differential equations for any one-dimensional inhomogeneity.

4. Inverse problem of geomagnetic sounding at ultra-low frequencies

The first problem is, in fact, one of the oldest and well-studied ill-posed problems. Here, we apply to this problem the new method of dual regularization to solve it in a rather new statement. In the ultra-low frequency band, analysis is much simplified. The approximation of the Leontovich's boundary conditions is mostly valid in this band, so the field in the medium can be considered as a plane wave with components E_x, H_y (further indices are omitted) of electric and magnetic field that propagate in the nadir direction. Also, the permittivity at low frequencies is determined by the conductivity σ as $\varepsilon = \varepsilon' + i\varepsilon'' \approx i4\pi\sigma/\omega$. Maxwell's equations for the complex amplitudes of electric and magnetic field are written as follows:

$$\frac{\mathrm{d}^2 E}{\mathrm{d}z^2} + i\frac{4\pi\sigma(z)\omega}{c^2} E = 0, \quad H = -i\frac{c}{\omega}\frac{\mathrm{d}E}{\mathrm{d}z}, \quad z_n \le z \le 0, \quad (29)$$

where $z_n < 0$ is the lower boundary of the analysis region that includes the inhomogeneity of the conductivity profile to be found. Fields are measured at the surface level $z = 0$ in dependence on frequency:

$$E(\omega, z = 0) = E_0(\omega), \; H(\omega, z = 0) = H_0(\omega), \quad \omega \in [\omega_1, \omega_2]. \quad (30)$$

These measurements are not independent, hence at the fixed electrical field, the spectral distribution of the corresponding magnetic field can be used to determine the profile of the medium conductivity. From (28) one can obtain the equivalent integral expression [9]:

$$H(\omega, z) = -\frac{4\pi i\omega}{c^2}\int_{-\infty}^{z}\left[\int_{-\infty}^{z'} H(\omega, z'')\mathrm{d}z''\right]\sigma(z')\mathrm{d}z', \tag{31}$$

and, assuming $z = 0$, obtain the non-linear integral equation to be solved to retrieve the conductivity depth profile $\sigma(z)$ by the measured spectrum of the near-surface magnetic field $H_0 = H(\omega, 0)$:

$$H_0(\omega) = -\frac{4\pi i\omega}{c^2}\int_{-\infty}^{0}\left[\int_{-\infty}^{z'} H[\sigma](\omega, z'')\mathrm{d}z''\right]\sigma(z')\mathrm{d}z'. \tag{32}$$

Similar equations can be obtained for the electric field. The Equation (31) can be solved iteratively beginning with the homogeneous profile $\sigma(z) = \sigma_0 = \text{const}$ in the kernel part of (32) as the first guess (Born approximation). For fields in homogeneous media, $E_0(\omega) = E_0^0(\omega)$, $H_0(\omega) = H_0^0(\omega)$, and one has exact formulas:

$$E^0(\omega, z) = E_0^0(\omega)\exp\left(\frac{z}{\delta_s} - i\frac{z}{\delta_s}\right), \tag{33}$$

$$H^0(\omega, z) = -\frac{(i+1)c}{\omega\delta_s}E_0^0(\omega)\left(\frac{z}{\delta_s} - i\frac{z}{\delta_s}\right) = H_0^0(\omega)\exp\left(\frac{z}{\delta_s} - i\frac{z}{\delta_s}\right), \tag{34}$$

where $\delta_s = c/\sqrt{2\pi\omega\sigma_0}$ is skin-depth. At that, the difference $\Delta H_0(\omega) = H_0(\omega) - H_0^0(\omega)$ can be used in the iterative algorithm:

$$\Delta H_0(\omega) = \int_{-\infty}^{0} K^i[\sigma_i](\omega, z')\Delta\sigma_{i+1}(z')\mathrm{d}z', \tag{35}$$

$$K^i[\sigma_i](\omega, z') = -\frac{4\pi i\omega}{c^2}\int_{-\infty}^{z'} H^i[\sigma_i](\omega, z'')\mathrm{d}z''$$

where the Fredholm integral equation of the first kind (35) is solved at each step of the iteration process using, for example, the method of generalized discrepancy developed for complex-valued functions in the Hilbert space W_2^1 (Sobolev's space) [14].

This algorithm has been studied in numerical simulation, and it was found that for strong inhomogeneities, typical in real conditions, retrieval errors were quite large. In such cases, the kernel of the equation calculated in the first guess has large deviations of the true kernel, and, as a result, the solution diverges from the real profile that leads to an increase of errors at next steps of the iteration process.

So, it is just the proper case to apply the proposed method of dual regularization. The mathematical statement of the inverse problem is based on the following assumptions: (i) the range of the inhomogeneity $\sigma(z)$ lies in the finite interval $z_n < z < 0$, $\sigma(z) = \sigma(z_n)$ at $z \le z_n$; (ii) values of fields E and H are finite at $z < z_n$. It gives the possibility to consider function classes determined in the finite interval $[z_n,\ 0]$, that makes it easier to argue the convergence of the regularization method. The solution of the direct problem is obtained from (29) to (30), so the statement of the inverse problem of geomagnetic sounding can be formulated like this: to find the conductivity profile $\sigma(z),\ z \in [z_n,\ 0]$ so that at any frequency $\omega \in [\omega_1, \omega_2]$ solutions $E[\sigma](\omega, z)$ and $H[\sigma](\omega, z)$ of the system (30) with boundary conditions

$$E[\sigma](\omega, z = 0) = E_0(\omega), \tag{36}$$

$$H[\sigma](\omega, Z = Z_n) = -(i + 1)\sqrt{2\pi\sigma_n/\omega}E[\sigma](\omega, z = z_n)$$

satisfy

$$H[\sigma](z = 0, \omega) = H_0(\omega). \tag{37}$$

The dual regularization method can be applied for fields' measurements as has been proposed in [16]. It should be noted that historically, beginning from the first works, the spectrum of the impedance $Z_0(\omega) = E_0/H_0$ has been used in analysis in the magnetotelluric exploration. It is easy to reformulate the problem in terms of impedance; however, to demonstrate results in the same way as in all three problems considered here, we involve in analysis the spectrum of the reflection coefficient $R_0(\omega)$ of equivalent vertical plane wave that can be calculated as $R_0(\omega) = (Z_0(\omega) - 1)/(Z_0(\omega) + 1)$, and the condition (30) is changed to

$$R[\sigma](\omega) = R_0(\omega). \tag{38}$$

In practice, magnetotelluric data are often measured in very noisy conditions. When we have multifrequency measurements, it is possible to minimize the noise like in [15], by the transformation of multifrequency data to time domain (by inverse Fourier transform over positive frequencies):

$$R_0(t) = \frac{1}{\Delta\omega}\int_{\omega} R_0(\omega)\exp(i\omega t)\mathrm{d}\omega, \tag{39}$$

where $\Delta\omega$ is the band of analysis. At that, the non-correlated part of noise is much suppressed in the range, where the signal is formed by subsurface inhomogeneities. Also, it is convenient to change the time parameter t to the space parameter that determines the effective depth of a scattering element $z_s = -0.5tc/\mathrm{Re}(\sqrt{\varepsilon})$ (taking into account the light velocity in a medium and signal path to and from a scattering element), and write the new condition for transformed data as follows:

$$R[\sigma](z_s) = R_0(z_s). \tag{40}$$

To apply the dual regularization in non-linear cases, it is necessary to use modified Lagrange functions with added penalty terms (13) taking into account that input data of the problem

(40) include a finite error $\delta_R > 0$. Assuming $l_1 = l_2 = 1$ in (13), the proper modified Lagrange function for our problem can be expressed as follows:

$$L_\mu^{\delta_R}[\sigma](\boldsymbol{\lambda}) = \|\sigma/\sigma_0\|^2 + \frac{1}{\Delta z_s}\int_{z_s} \langle \boldsymbol{\lambda}(z_s), \mathbf{R}[\sigma](z_s) - \mathbf{R_0}(z_s)\rangle \mathrm{d}z_s$$
$$+ \mu\left\{ \left(\frac{1}{\Delta z_s}\int_{z_s} \left|\mathbf{R}[\sigma](z_s) - \mathbf{R_0}(z_s)\right|^2 \mathrm{d}z_s\right)^{1/2} + \frac{1}{\Delta z_s}\int_{z_s} \left|\mathbf{R}[\sigma](z_s) - \mathbf{R_0}(z_s)\right|^2 \mathrm{d}z_s \right\} \quad (41)$$

where $\langle\cdot\rangle$ is a scalar product, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, $\|\sigma/\sigma_0\|^2_{L_2} = \frac{1}{\Delta z}\int_{\Delta z} (\sigma(z)/\sigma_0)^2 \mathrm{d}z$, Δz is the region of analysis, $\sigma_0 = \sigma(z = 0)$, $\mathbf{R} = (\mathrm{Re}R, \mathrm{Im}R)$ are the two-dimensional vectors, $\mu > 0$. When the parameter μ is large enough, the minimum of the modified Lagrange function $L_\mu^{\delta_R}[\sigma](\boldsymbol{\lambda})$ over $\sigma(z)$ exists for certain for any $\boldsymbol{\lambda}$. The regularized dual problem is a problem of maximization of the concave functional (14) in the Hilbert space $L_2{}^2(z_{s1}, z_{s2})$. In this case, it is expressed as follows:

$$W_\mu^{\delta_R}(\boldsymbol{\lambda}) = V_\mu^{\delta_R}(\boldsymbol{\lambda}) - \alpha||\boldsymbol{\lambda}||^2 = \min_{\sigma\in D} L_\mu^{\delta_R}[\sigma](\boldsymbol{\lambda}) - \alpha||\boldsymbol{\lambda}||^2 \to \max_{||\lambda||\le\mu}, \quad (42)$$

where the maximum is found over $\boldsymbol{\lambda}$ from the set $\Lambda_\mu \equiv \{\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in L_2{}^2(z_{s1}, z_{s2}): ||\boldsymbol{\lambda}|| \le \mu\}$, $D = \{\sigma \in L_2(z_n, 0): 0 \le \sigma(z) \le \sigma_{\max}\}$, where $\sigma_{\max}$ is a sufficiently large value. At the maximization of the functional (41), we use an element of the supergradient (15) in this functional that can be expressed explicitly:

$$\partial W_\mu^{\delta_R}(\boldsymbol{\lambda}) = \mathrm{cl\ conv}\{\mathbf{R}[\sigma] - \mathbf{R}_0 : \sigma \in \Sigma_\mu[\boldsymbol{\lambda}]\} - 2\alpha\boldsymbol{\lambda}, \quad (43)$$

where cl conv A means the closing convex hull of a set A, $\Sigma_\mu[\lambda] = \mathrm{Argmin}\{L_\mu[\sigma](\lambda): \sigma \in D\}$. The desired solution is obtained as a saddle point of this process of minimization of (41) over σ and simultaneous maximization of (42) over dual variable $\boldsymbol{\lambda}$.

The numerical algorithm that realizes this method is based on conventional gradient minimization of (41) and maximization of the functional (42) using the expression for its gradient (43). This problem should be solved for data with a finite error, so that $||\mathbf{R}[\sigma(z)] - \mathbf{R}_0||^2_{L_2} \equiv \frac{1}{\Delta z_s}\int_{z} \left|\mathbf{R}[\sigma(z)](z_s) - \mathbf{R}_0(z_s)\right|^2 \mathrm{d}z_s \le \delta_R^2$, where Δz_s determines the region of data used in analysis. Based on reasons to develop a rather universal algorithm and results of numerical simulations, stopping rules for developed algorithms are chosen somewhat different from the theoretical rule [22,23,38] given after formula (19). At that, we assume $\mu = 10$ in (41), and begin the iterative procedure (18) with $\boldsymbol{\lambda}^{k=1} = 0$, $\partial W_\mu^{\alpha(k=1)}(\lambda) = \frac{1}{\Delta z_s}\{\mathbf{R}[\sigma^{k=1}(z) = \sigma_0](z_s) - \mathbf{R}_0(z_s)\} - 2\alpha\boldsymbol{\lambda}$, and calculate the initial discrepancy $\tilde{\delta}^{k=1} = \left\|\mathbf{R}[\sigma^{k=1}(z) = \sigma_0] - \mathbf{R}_0\right\|_{L_2}$. In the iterative procedure (18) further values of discrepancy $\tilde{\delta}^k = \left\|\mathbf{R}[\sigma^k(z)] - \mathbf{R}_0\right\|_{L_2}$ are calculated; at that, we used sequences $\alpha^k = k^{-1/3}$, $\beta^k = 10^{-2}k^{-1/2}$, $k = 1, 2, \ldots$ that satisfy conditions (19). This iterative procedure (proceeds up to the largest number $k = k(\delta_R)$, for which one of stopping rules $\left\|\partial W_\mu^{\delta_R}(\lambda^k)\right\| > a, \tilde{\delta}^k \ge b\delta_R$ is fulfilled (a, b are values determined from the numerical simulation). The corresponding point σ^k gives us the desired solution of the problem.

The computer algorithm of this method has been worked out and used in numerical simulations of this inverse problem. For initial profile $\sigma(z)$, the frequency dependence $R_0(\omega)$ is calculated for the vertical incidence; random Gauss-distributed errors are added; these

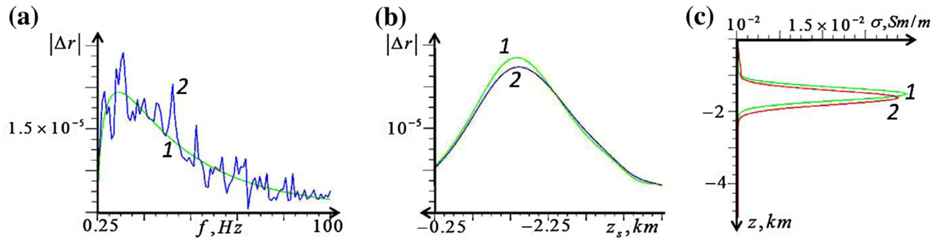

Figure 1. (*a*) 1 – $\Delta r(f) = |R_0[\sigma(z)](f) - R[\sigma_0](f)|$ calculated for initial profile (line 1 in Figure 1(*c*)), 2 – 'data of measurements' with random errors; (*b*) 1 – synthesized pseudopulse $\Delta r(z_s) = |R_0[\sigma(z)](z_s) - R[\sigma_0](z_s)|$ obtained for $\Delta r(f)$ (line 1 in Figure 1(*a*)), 2 – synthesized pseudopulse $\Delta r(z_s)$ obtained for data with errors (line 2 in Figure 1(*a*)); (*c*) 1 – initial profile $\sigma(z)$; 2 –retrieved profile; $f = \omega/2\pi$.

'data of measurements' are transformed to $R_0(z_s)$ that is used as input data in (39). Finally, the retrieved profile is compared to the initial one. It should be mentioned that the conductivity of the earth crust lies in the very broad range 10^{-5}–10^{-1} Sm/m, so it is necessary to adjust parameters of measurements to available conditions in each case specifically.

The example of numerical simulation shown in Figure 1 demonstrates the retrieval of the conductivity profile $\sigma(z)$ in the medium with a high enough conductivity $\sigma_0 = \sigma(z = 0) = 0.01$ Sm/m. To extract the informative part of the signal, input data are given as deviations $\Delta r = |R_0[\sigma(z)] - R[\sigma_0]|$ of reflection coefficients calculated for the inhomogeneous medium with $\sigma(z)$ from those calculated for the half-space with $\sigma = \sigma_0,\ z \le 0$.

In Figure 1(a), it is possible to see calculated multifrequency data and simulated 'measurement data' with random normally distributed uncorrelated errors with rms = 5×10^{-6}. The skin-depth δ_s is changed in the interval 0.1–10 km in the frequency range of analysis. In Figure 1(b), corresponding synthesized pseudopulses are shown, and it can be seen that errors are much suppressed at such a transformation. Dependence $\Delta r(z_s)$ that corresponds to 'measurements' (line 2) is used in the dual regularization algorithm (40–43) to retrieve a sharp gauss inhomogeneity at the depth 1.5 km (see in Figure 1(c)). Results demonstrate a good retrieval at the extremely high level of data errors. It is interesting to compare the position z_{smax} of the maximum of synthesized pulse in the distribution $\Delta r(z_s)$ (Figure 1b) to the position z_{max} of the maximum in depth profile $\sigma(z)$ in Figure 1(c). It is easily seen that $z_{smax} \approx z_{max}$. This fact can be used for the preliminary diagnostics of the underground conductivity stratification.

The simulation shown in Figure 2 demonstrates an example of retrieval in the medium with much less conductivity $\sigma_0 = 2 \times 10^{-4}$ Sm/m. In Figure 2(a), again, it is possible to see calculated multifrequency data and simulated 'measurement data' with random errors (rms = 5×10^{-4}).

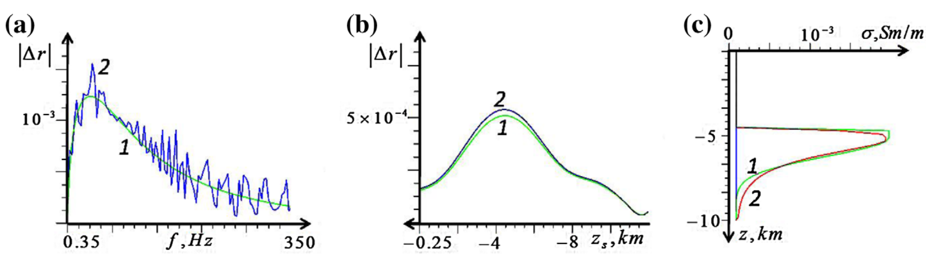

Figure 2. The same notation as in Figure 1.

In Figure 2(b) corresponding synthesized pseudopulses are shown, and $\Delta r(z_s)$ that corresponds to 'measurements' (line 2) is used to retrieve the simulated inhomogeneity of a more complicated form: a combination of a layer with the constant conductivity with a half of gauss profile at the depth of about 5 *km* (Figure 2(c)). Results demonstrate the ability of the developed algorithm to retrieve details of subsurface profiles. Again, it is worth mentioning that the position of the maximum in dependence $\Delta r(z_s)$ corresponds to that of the maximum of $\sigma(z)$.

Notwithstanding these impressive results, it should be stressed that this method doesn't work automatically – it must be adjusted to each case specifically, including the choice of measurement and algorithm parameters, schemes of functional minimizations, control of discrepancy convergence up to the level of measurement errors, numerical simulation and so on. Also, it is necessary to note that in cases when errors are not uncorrelated, i.e. have the frequency dependence similar to the real signal, it leads to large distortion of retrieved parameters; however, they should be studied in every case separately.

5. Microwave monitoring of water diffusion in soil

Our second problem is a new one and much more complicated in numerical realization. The study in this case is based not only on numerical simulation, but also on experimental testing. In experiments, we have applied equipment of near-field microwave tomography of subsurface dielectric inhomogeneities developed in [13–15], for monitoring one-dimensional process of water diffusion in soil. The main motivation of this study was to test this new method of dual regularization for this simpler, one-dimensional inverse scattering problem to overcome available restrictions related to the inapplicability of the Born approximation for strong inhomogeneities. However, even this simplified problem was extremely complicated for calculations, and we used the code parallelization in the numerical algorithm to solve this problem with the supercomputer cluster in Nizhny Novgorod State University.

The near-field tomography of three-dimensional distribution of complex permittivity $\varepsilon_1(\mathbf{r})$ is based on measurements of signal complex amplitudes at 801 frequencies in the range 1.7–7.0 GHz with the source–receiver system composed of Agilent E5071B vector network analyzer and two identical transmitting and receiving planar bow-tie antennas in the bistatic configuration. They were scanning together in the rectangle x–y area above the targets.

In the following analysis, like in [15], we use the plane wave decomposition and the transformation of multifrequency data to time domain. Variations of complex amplitudes of the received signal in the point $\mathbf{r}_r$ (the vector that marks the receiving antenna position) can be expressed as the inverse Fourier transform of their transversal spectrum

$$s(x_r, y_r, \omega) = \iint s(\kappa_x, \kappa_y, \omega)\exp(i\kappa_x x_r + i\kappa_y y_r)\mathrm{d}\kappa_x \mathrm{d}\kappa_y. \tag{44}$$

For plane antennas, this signal is the convolution of scattered field variations and antenna function, so one has its k-space representation as follows:

$$s(\kappa_x, \kappa_y, z_r) = 4\pi^2 \mathbf{E}_1(\kappa_x, \kappa_y, z_r)\mathbf{F}(\kappa_x, \kappa_y). \tag{45}$$

The scattered field is determined by the incident field and the reflection coefficients on TH and TE polarizations:

$$\mathbf{E}_1(\kappa_x,\kappa_y,z_r)=[R^{\parallel}(\kappa_x,\kappa_y)\mathbf{E}_0^{\parallel}(\kappa_x,\kappa_y,0)+R^{\perp}(\kappa_x,\kappa_y)\\ \mathbf{E}_0^{\perp}(\kappa_x,\kappa_y,0)]\exp\left(\sqrt{k^2-\kappa_x^2-\kappa_y^2}z_r\right). \tag{46}$$

where k-space representation for the incident field obtained in [14] is:

$$\begin{aligned}\mathbf{E_0}(\kappa_x,\kappa_y,0)\ &=-\tfrac{2\pi}{\omega}\exp\left\{i\sqrt{k^2-\kappa_x^2-\kappa_y^2}z_s\right\}\\ &\times\left\{j_x(\kappa_x,\kappa_y,z_s)\left\{\left[\tfrac{\kappa_x^2k_z}{\kappa_\perp^2}\vec{x}_0+\tfrac{\kappa_x\kappa_yk_z}{\kappa_\perp^2}\vec{y}_0-\kappa_x\vec{z}_0\right]_{\parallel}+\tfrac{k^2}{\kappa_\perp^2k_z}\left[\kappa_y^2\vec{x}_0-\kappa_x\kappa_y\vec{y}_0\right]_{\perp}\right\}\right.\\ &\left.+j_y(\kappa_x,\kappa_y,z_s)\left\{\left[\tfrac{\kappa_x\kappa_yk_z}{\kappa_\perp^2}\vec{x}_0+\tfrac{\kappa_y^2k_z}{\kappa_\perp^2}\vec{y}_0-\kappa_y\vec{z}_0\right]_{\parallel}+\tfrac{k^2}{\kappa_\perp^2k_z}\left[-\kappa_x\kappa_y\vec{x}_0+\kappa_x^2\vec{y}_0\right]_{\perp}\right\}\right\},\end{aligned} \tag{47}$$

$$\begin{aligned}j_i(\kappa_x,\kappa_y,z_s)\ &=\tfrac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}j_i\left(x_r-\delta x-x,y_r-\delta y-y,z_s\right)e^{-i\kappa_xx-i\kappa_yy}\mathrm{d}x\mathrm{d}y\\ &=\tfrac{1}{4\pi^2}e^{-i\kappa_xx_r-\delta xx-i\kappa_yy_r-\delta yy}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}j_i(x',y',z_s)e^{-i\kappa_xx'-i\kappa_yy'}\mathrm{d}x\mathrm{d}y\\ &=\tfrac{1}{4\pi^2}e^{-i\kappa_x(x_r-\delta x)-i\kappa_y(y_r-\delta y)}j_i(\kappa_x,\kappa_y),\end{aligned} \tag{48}$$

where $k=\omega/c$, $k_z=\sqrt{k^2-\kappa_x^2-\kappa_y^2}$, $\kappa_\perp=\kappa_x^2+\kappa_y^2$, z_s, z_r are source and receiver altitudes above the surface. The current distribution on antennas and its transversal spectrum have been calculated and presented in [15]. Also, we assume the reciprocity condition $F_i(\kappa_x,\kappa_y,z_s)=\text{const}\ j_i(\kappa_x,\kappa_y)$. As a result, we obtain the expression for the signal measured as a function of time t:

$$\begin{aligned}s(x_r,y_r,\omega,t)\ &=-\text{const}\iint\tfrac{1}{2\pi\omega}\exp\left\{i(\kappa_x\delta x+i\kappa_y\delta y+\sqrt{k^2-\kappa_x^2-\kappa_y^2}(z_s+z_r)\right\}\\ &\times[R^{\parallel}[\varepsilon(z,\omega,t)](\kappa_x,\kappa_y)\left\{j_x^2\tfrac{\kappa_x^2k_z}{\kappa_\perp^2}+j_y^2\tfrac{\kappa_y^2k_z}{\kappa_\perp^2}\right\}\\ &+R^{\perp}[\varepsilon(z,\omega,t)](\kappa_x,\kappa_y)\left\{j_x^2\tfrac{k^2\kappa_y^2}{\kappa_\perp^2k_z}+j_y^2\tfrac{k^2\kappa_x^2}{\kappa_\perp^2k_z}\right\}]\mathrm{d}\kappa_x\mathrm{d}\kappa_y.\end{aligned} \tag{49}$$

The value of const in (48) is determined by measurements of initially dry sandy ground.

So, to calculate (48), it is enough to calculate reflection coefficients for plane waves reflected from the half-space with the one-dimensional profile of permittivity. Here, like in [15], we use the possibility to transform the multifrequency problem into that in time domain τ using the inverse Fourier transformation of multifrequency data to the synthesized pseudopulse

$$s(\tau,t)=\frac{1}{\Delta\omega}\int_{\omega}s(x_r,y_r,\omega,t)\exp(i\omega\tau)\mathrm{d}\omega \tag{50}$$

that can be represented in dependence on the effective depth parameter z_s according $s(z_s,t)=s(\tau=-2z_s\mathrm{Re}\sqrt{\varepsilon_0}/c,t)$. The strong maximum of $s(z_s)$ marks the position of the surface; values of z_s are counted from this point.

There is a problem that reflection coefficients in (49) are determined by the frequency-dependent complex permittivity depth profile $\varepsilon(z,\omega)$, and this two-dimensional parameter is unsuitable for retrieval. In this case, the proper one-dimensional parameter to be retrieved is the profile of volume water content $f_w(z)$ in the sand that determines permittivity variation in the process of water diffusion in soil. Various dielectric mixing formulas can be applied to calculate the permittivity profile; we use the De Loor formula [45] that determines $\varepsilon(z,\omega) = F[f_w(z)](\omega)$. Then, our problem is formulated like this: to find the profile variations $f_w(z, t)$ that satisfy the equation between calculated and measured data:

$$s[f_w(z)](z_s, t)) = s_0(z_s, t). \tag{51}$$

So, according to (13), the modified Lagrange function for the dual regularization method can be written at each t as follows:

$$L_\mu^{\delta_R}[f_w](\boldsymbol{\lambda}) = \|f_w\|^2 + \frac{1}{\Delta z_s}\int_{z_s} \left\langle \boldsymbol{\lambda}(z_s), \left(\mathbf{s}[f_w](z_s) - \mathbf{s_0}(z_s)\right)\right\rangle \mathrm{d}z_s$$
$$+ \mu\left\{\left(\frac{1}{\Delta z_s}\int_{z_s} \left|\mathbf{s}[f_w](z_s) - \mathbf{s_0}(z_s)\right|^2 \mathrm{d}z_s\right)^{1/2} + \frac{1}{\Delta z_s}\int_{z_s} \left|\mathbf{s}[f_w](z_s) - \mathbf{s_0}(z_s)\right|^2 \mathrm{d}z_s\right\}, \tag{52}$$

where $\|f_w\|_{L_2}^2 = \frac{1}{\Delta z}\int_{\Delta z} f_w(z)^2 \mathrm{d}z, \boldsymbol{\lambda} = (\lambda_1, \lambda_2), \mu > 0$. The regularized dual problem is a problem of maximization of the concave functional

$$W_\mu^{\delta_R}(\boldsymbol{\lambda}) = \min_{\sigma \in D} L_\mu^{\delta_R}[f_w](\boldsymbol{\lambda}) - \alpha||\boldsymbol{\lambda}||^2 \to \max_{||\lambda|| \le \mu}, \tag{53}$$

where $D = \{f_w \in L_2(z_n, 0){:}0 \le f_w(z) \le 1\}$. Here, like in (40–42), the complex-valued reflection signal **s** is considered as a two-dimensional vector. The supergradient of the functional (53) is expressed explicitly, similar to (43). The desired solution, is obtained as a saddle point of the process of minimization of (52) over f_w at the maximization of (53) over the dual variable $\boldsymbol{\lambda}$, using the same algorithm as described above for the problem of geomagnetic sounding (40)–(42). To retrieve the whole water diffusion process $f_w(z, t)$, the inverse problem (50)–(52) should be solved at each time t.

In our numerical study, we simulated conditions, similar to those in real experiment. The evolution of the water content in the diffusion process has been simulated by exponential profiles $f_w(z, t) = f_{w0}\exp\,[zt_0/\Delta z(t_0 + t)]$. For each time t the corresponding signal $s(\omega, t)$ has been calculated as a function of frequency; random errors that correspond to those in real experiment (they are frequency-uncorrelated in this case) are added, and these 'measurement data' are transformed into a synthesized pseudopulse $s_0(z_s, t)$ that is used to solve the inverse problem.

In Figure 3, one can see simulated and retrieved profile evolution; in Figure 4 – the corresponding evolution of pseudopulse calculated by exact $s(\omega, t)$ (in Figure 3(a)), and by data with errors (in Figure 3(b)).

As it is possible to see from Figure 3, the simulated evolution is retrieved quite well.

In real experiment, the initial water content was prepared by uniform water spilling under source–receiver antennas – totally about 1 g/cm^2. After its absorption into the soil near-surface layer, measurements begin.

In Figure 5(a), one can see the dynamics of measured multifrequency data related to water diffusion; the corresponding dynamics of pseudopulse synthesized by measured multifrequency data is given in Figure 5(b), and the retrieved evolution of the water volume content $f_w(z, t)$ in the process of water diffusion in sand is demonstrated in Figure 5(c). Effects of

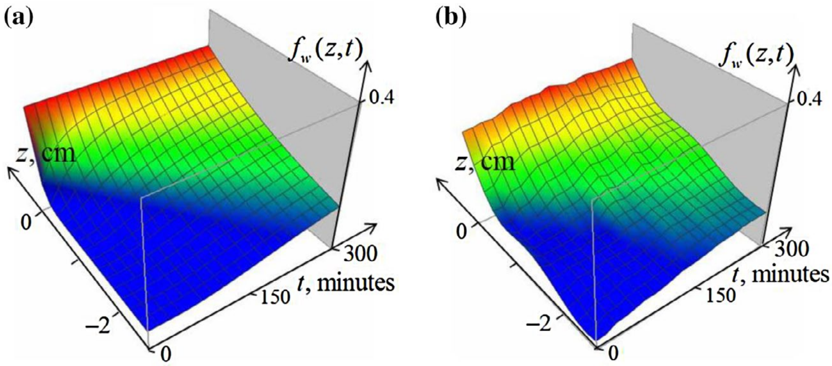

Figure 3. Simulation of water diffusion retrieval: (*a*) simulated $f_w(z, t)$; (*b*) retrieved by data shown in Figure 4(*b*).

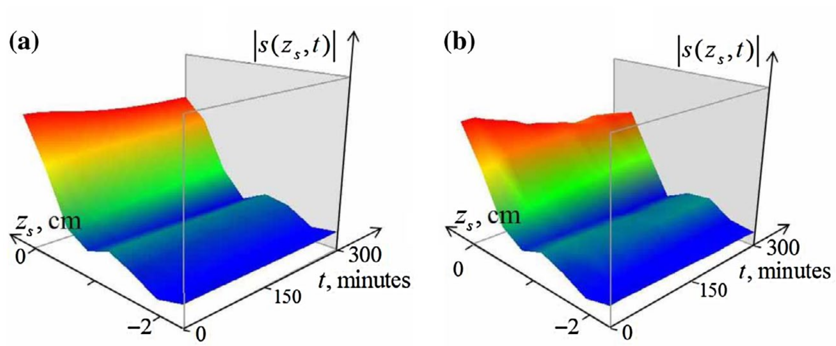

Figure 4. Dynamics of pseudopulse: (*a*) calculated by exact 'data'; (*b*) calculated by data with errors.

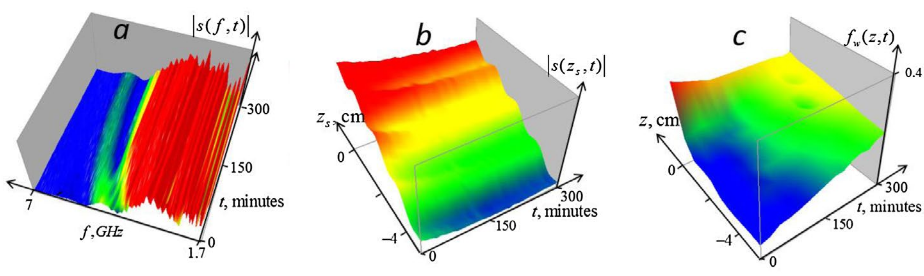

Figure 5. (*a*) Measured dynamics of multifrequency data in the process of water diffusion in sandy soil; (*b*) dynamics of pseudopulse synthesized by data in Figure 5(*a*); (*c*) retrieved evolution of water volume content in this process obtained from the solution of (50)–(52).

the diffusion process are seen both in measured multifrequency data (Figure 5(a)) and in corresponding synthesized pseudopulse (Figure 5(b)). As it is possible to see in Figure 5(c), the retrieved evolution of water content looks quite reasonable: at deeper layers it grows with time, and the profile tends to the homogeneous distribution with the content that is near to its saturated value (about 0.3). We hope that a similar dual regularization approach can be applied in the much more difficult three-dimensional problem of subsurface microwave tomography.

6. Diagnostics of periodic inhomogeneity of multilayer structures by multifrequency X-ray reflection

The third problem considered demonstrates the adaptation of the dual regularization method to the diagnostics based on multifrequency measurements of the power reflection coefficient, when the phase information is lost. Here, we study the diagnostics of permittivity inhomogeneities in multilayer periodical structures that are basic elements of the modern X-ray optics.[12] Some deviations from the desired perfect meander structure appear as a result of the material diffusivity related to the epitaxial technique used in the production of structures. One-dimensional structure defects can be described in terms of the periodic permittivity profile. Multilayered periodical structures in X-ray optics are widely used as reflectors, polarizers and filters in the 'soft' X-ray range since their invention in 1976.[46] Their parameters are optimized for different purposes, but deviations from desired meander structures distort their predicted properties. For diagnostics of these structures, it is possible to use reflectometry measurements of X-ray scattering. The reflectometry method has obvious advantages: it is non-contact, non-destructive and fast in comparison with the electron microscopy or secondary ion mass spectrometry.

Unfortunately, unlike two above-considered problems, phase measurements are very complicated in this spectral band, so power reflection coefficients are used in analysis that makes the solution much more difficult, because such a problem is more underdetermined, i.e. more ill-posed. The problem of the permittivity profile evaluation from the X-ray scattering data has been considered earlier in frameworks of parameterization approaches. In the theory represented in [47], a symmetrical meander structure with exponential inhomogeneities has been considered. In [48], authors proposed the linear model to estimate the asymmetry of the profile. In [12], this inverse problem has been solved to obtain the permittivity profile of inhomogeneities from the obtained integral equation. But, as it was found, the solution of this problem based on non-linear integral equation also has serious restrictions related to errors of the Born approximation used as the first guess in the iterative solution and to errors of such integral representation itself. Again, to overcome these problems, algorithms of the dual regularization method have been worked out and studied in numerical simulation.[49] It was shown that profiles of diffuse inhomogeneities can be retrieved with a good quality, but the success was strongly dependant on the successful choice of the first guess. The numerical study shows that the absence of phase information leads to serious problems in the solution. Sometimes results deviate far from the exact solution, i.e. it corresponds to a local minimum of the Lagrange functional. To realize all advantages of the dual regularization, it was necessary to fit the parameters of the iteration scheme using the available freedom of their choice, or to find the solution as the deviation of a reasonably chosen first guess.

Here, to compensate the lack of phase information, we propose a new approach in the dual regularization method that involves a priori information about the membership of the desired solution to a compact set of functions, in particular – to bound monotonous or convex monotonous functions.

It is important to note that measurement accuracy high enough is needed to apply this method in practice. For example, to achieve the necessary level of accuracy, multiple measurements have been averaged in [12] over 10 realizations. To avoid this difficulty, we propose to use the transformation of multifrequency data to that in time domain in the same way as above. In such a synthesized pulse, measurement errors are much suppressed in the informative part of this pseudopulse.

Following [12], consider a periodic multilayer (in z-direction) medium with the period $d = d_1 + d_2$ with a complex permittivity profile $\varepsilon(z) = \varepsilon'(z) + i\varepsilon''(z)$. Assuming that this profile of inhomogeneities $\varepsilon_1(z) = \varepsilon_1(z + d)$ is also periodic, it can be expressed as follows:

$$\varepsilon(z) = \begin{cases} \varepsilon_{01}, & z < 0 \\ \varepsilon_{02} + \varepsilon_1(z), & id \le z < id + d_1 \\ \varepsilon_{03} + \varepsilon_1(z), & id + d_1 \le z \le id + d_1 + d_2 \\ \varepsilon_{04}, & z > Nd, \end{cases} \quad (54)$$

$i = 0, 2, \ldots, N - 1$. Dielectric parameters of layers are, in general, absorbing and frequency-dependent. Because $\varepsilon_1(z)$ is formed by mutual penetration of two components of the meander structure, it is reasonable to represent it as

$$\varepsilon_1(z) = \begin{cases} \frac{\varepsilon_{02}+\varepsilon_{03}}{2} + \frac{\varepsilon_{02}-\varepsilon_{03}}{2} f(z), & id \le z < id + d_1 \\ \frac{\varepsilon_{02}+\varepsilon_{03}}{2} - \frac{\varepsilon_{02}-\varepsilon_{03}}{2} f(z), & id + d_1 \le z \le id + d_1 + d_2 \end{cases}, \quad (55)$$

where complex-valued permittivity perturbations of this mixture are determined by the real-valued profile$f(z)$ (of course, any other mixing formula can be used here).

In the proposed reflectometry diagnostics, the frequency spectrum of the difference

$$\Delta r_0(\omega) = |R_m(\omega)|^2 - |R_0(\omega)|^2 \quad (56)$$

between the measured power reflection coefficient and that calculated by known parameters of the ideal meander structure with parameters d_1, d_2, N is in use. The statement of the inverse scattering problem is formulated like this: to find such a profile $f(z)$ that the condition

$$\Delta r[f](\omega) = |R[f](\omega)|^2 - |R_0(\omega)|^2 = \Delta r_0(\omega) \quad (57)$$

for reflection coefficients $|R|^2[f]$ calculated for a profile $f(z)$ is satisfied at any frequency $\omega \in [\omega_1, \omega_2]$.

As it was mentioned above, the real measurement errors can be comparable to values of Δr_0. To avoid this difficulty, we use the transformation of multifrequency data to time domain, introducing, as above, the length parameter z_s that in this case, of course, doesn't have the simple meaning of the depth of scattering elements:

$$\Delta r_0(z_s) = \frac{1}{\Delta\omega} \int_\omega \Delta r_0(\omega) \exp(i\omega z_s/c) \mathrm{d}\omega. \quad (58)$$

At that, the condition (56) is transformed to

$$\Delta r[f](z_s) = |R[f](z_s)|^2 - |R_0(z_s)|^2 = \Delta r_0(z_s). \quad (59)$$

The modified Lagrange functional (13) of this problem can be written as follows:

$$L_{\mu}^{\delta_R}[f](\lambda) = ||f||^2 + \frac{1}{z_{s\max}}\int_0^{z_{s\max}} \lambda(z_s)\mathrm{Re}\left(\Delta r[f](z_s) - \Delta r_0(z_s)\right)\mathrm{d}z_s$$
$$+ \mu\left\{\left(\frac{1}{z_{s\max}}\int_0^{z_{s\max}} \left|\Delta r[f](z_s) - \Delta r_0(z_s)\right|^2 \mathrm{d}z_s\right)^{1/2} + \frac{1}{z_{s\max}}\int_0^{z_{s\max}} \left|\Delta r[f](z_s) - \Delta r_0(z_s)\right|^2 \mathrm{d}z_s\right\}. \tag{60}$$

where $\|f\|_{L_2}^2 = \frac{1}{\Delta z}\int_{\Delta z} f(z)^2 \mathrm{d}z$. In this case, real and imaginary parts of the transformed real-valued reflection coefficient carry the same information, so only the real-valued term is included in the Lagrange functional. The corresponding regularized modified dual problem consists of maximization of the concave functional:

$$W_{\mu}^{\delta_R}(\lambda) = \min_{f\in D} L_{\mu}^{\delta_R}[f](\lambda) - \alpha\|\lambda\|^2 \to \max, \tag{61}$$

The supergradient of the functional (61) is expressed explicitly, similarly to (43). Thus, as above, the scheme of the dual regularization method consists of the gradient minimization of (60) over *f* at the simultaneous maximization of (61) over λ using the same iterative scheme as described above for the problem (40)–(42). The function *f*(*z*) in the saddle point gives us the desired regularized solution.

The a priori information about whether the desired solution *f* belongs to one of compact sets of functions *D* (monotonously decreasing $M_{\downarrow c}$ or increasing $M_{\uparrow c}$ functions bound by a finite constant, or convex monotonously decreasing $\hat{M}_{\downarrow c}$ or increasing $\hat{M}_{\uparrow c}$ functions) is introduced at the transformation of infinite-dimensional problem (59)–(61) to a finite-dimensional one. There are four regions of the monotonous increasing and decreasing over a structure period:

$$D = f \in \left\{\left\{\begin{array}{ll} M_{\uparrow c} \text{ or } \hat{M}_{\uparrow c}, & 0 \le z < d_1/2 \\ M_{\downarrow c} \text{ or } \hat{M}_{\downarrow c}, & d_1/2 \le z < d_1 \\ M_{\uparrow c} \text{ or } \hat{M}_{\uparrow c}, & d_1 \le z \le d_1 + d_2/2 \\ M_{\downarrow c} \text{ or } \hat{M}_{\downarrow c}, & d_1 + d_2/2 \le z \le d_1 + d_2 \end{array}\right\} \in L_2(0,d): 0 \le f(z) \le 1\right\}. \tag{62}$$

At a given discretization *n* = *N/4*, monotonous parts of function *f* in each of four regions in (62) can be approximated as corresponding convex combinations

$$f = \sum_{j=0}^{n} a_j T^{(j)}, \tag{63}$$

$$T^{(j)} = \left\{\begin{array}{ll} 1, & i \le j \\ 0, & i > j \end{array}\right., \quad f \in M_{\downarrow c}, \quad T^{(j)} = \left\{\begin{array}{ll} 1, & i \le j \\ \frac{n-i}{n-j}, & i > j \end{array}\right., \quad f \in \hat{M}_{\downarrow c}$$

$$T^{(j)} = \begin{cases} 0, & i < j \\ 1, & i \geq j \end{cases}, \quad f \in \mathrm{M}_{\uparrow c}, \quad T^{(j)} = \begin{cases} \frac{n-j}{n-i}, & i < j \\ 1, & i \geq j \end{cases}, \quad f \in \hat{M}_{\uparrow c}, \quad T^{(1)} = 0$$

of properly chosen n-dimensional vectors $T^{(j)}$ (basis in R^n) that correspond to vertexes of corresponding convex polyhedrons [11] that represent these sets. So, in the gradient minimization of (60), instead of vector f, we deal with vector of coefficients a in the decomposition (63), and it gives us a monotonous function f at each step of this process.

Numerical algorithms of the dual regularization method (59–61) have been worked out, using the same iterative scheme as above, and applied in the simulation of the proposed diagnostics of inhomogeneity profile of permittivity in multilayer structures. Here, we present the numerical simulation for inhomogeneities in the periodic Mo-Si 50-layer structure (the same as in [12]), which is retrieved by multifrequency reflectometry data in the wavelength range $\lambda = 12.5 \div 14.5$ nm at the elevation angle $\theta = 85°$. As it has been shown in [12], in this spectral range the reflection coefficient has a considerable sensitivity to profile variations.

In Figure 6, results of simulation of dual regularization method (59–60) on the sets $\hat{M}_{\downarrow c}$ and $\hat{M}_{\uparrow c}$ are shown. In this case, we use multifrequency 'measurement data' with errors, which are simulated by uncorrelated Gauss-distributed random distribution with rms = 0.036 (Figure 6(a)) that correspond to the accuracy of single measurements in [12], where, to achieve an acceptable error level, data have been averaged using 10 independent measurements (errors were frequency-uncorrelated).

One can see that the 'measurement' noise in Figure 6(a) (line3) is mainly suppressed in the corresponding pseudopulse in Figure 6(b) in the whole region of analysis, because it is shifted to larger values of z_s. As a result, a good accuracy of retrieval is achieved as is seen in Figure 6(c). Profiles of inhomogeneity are retrieved quite well notwithstanding the loss of the phase information. It is hardly possible to realize a universal algorithm suitable for an arbitrary profile of inhomogeneity; nevertheless, for diffusive inhomogeneities, when profiles are monotonously decreasing (increasing) from layers' interfaces, the developed algorithm gives good results in a wide enough range of possible parameters of inhomogeneities.

To demonstrate advantages of the data transformation more in details, the informative part of the signal $|\Delta r| = \left| |R|^2 - |R_0|^2 \right|$ is given along with the noise $|\delta r|$ in Figure 7(a); their transformations in synthesized pseudopulses are shown in Figure 7(b).

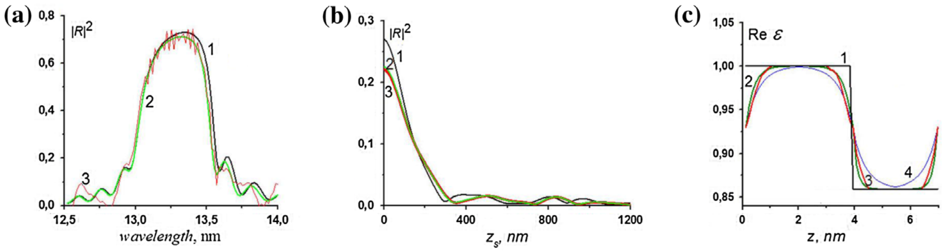

Figure 6. (*a*) 1 – reflection coefficient $|R_0(\lambda)|^2$ for unperturbed structure, 2 – $|R(\lambda)|^2$ for simulated profile, 3 – data with errors $|R_m(\lambda)|^2$; (*b*) corresponding synthesized pseudopulses: 1 – $|R_0(z_s)|^2$, 2 – $|R(z_s)|^2$, 3 – $|R_m(z_s)|^2$; (*c*) 1 – perfect periodic meander structure (one period), 2 – simulated profile of Re$\varepsilon(z)$, 3 – retrieved profile, 4 – first guess.

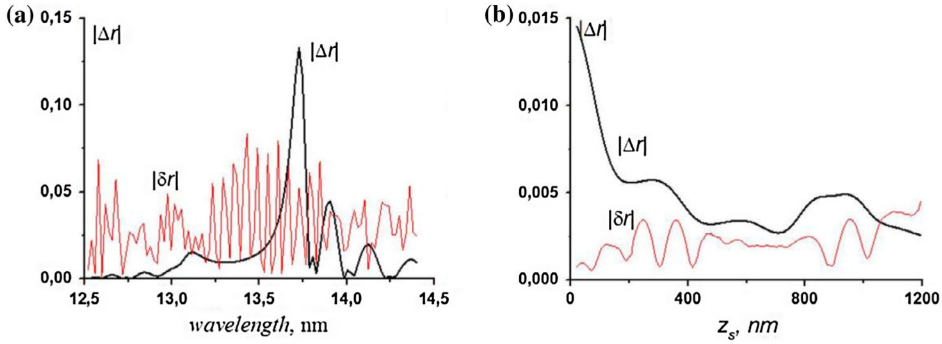

Figure 7. (*a*) Informative part of signal $|\Delta r(\lambda)| = \left||R(\lambda)|^2 - |R_0(\lambda)|^2\right|$ for simulated profile and the random error $|\delta r(\lambda)|$; (*b*) corresponding synthesized pulses $|\Delta r(z_s)| = \left||R(z_s)|^2 - |R_0(z_s)|^2\right|$, $|\delta r(z_s)|$ vs. z_s.

As it is clearly seen from Figure 7, large measurement errors in measured multifrequency data (Figure 7(a)) are much suppressed up to large values of delay parameter $z_s > z_{smax}$ in the corresponding synthesized pseudopulse (Figure 7(b)), so we have an informative interval broad enough, where the signal exceeds the level of errors. Using values of the pseudopulse in this interval, a good retrieval quality has been obtained as it is demonstrated in Figure 6.

The described numerical simulation of the proposed method of dual regularization in the pseudopulse diagnostics of periodic structures mostly leads to a good retrieval for profiles with various gradients of permittivity. But again, it should be stressed that this method doesn't work automatically; in cases, when discrepancy can't be minimized below the level of data errors, the initial guess should be changed and obtained solutions always should be studied attentively in the numerical simulation.

7. Conclusion

Results of this study show that the dual regularization method provides a remarkable progress in the solution of above-considered non-linear ill-posed problems. It is demonstrated that this new method can be used in the vast range of applications to physical diagnostics in geophysical prospection, remote sensing and non-destructive testing. Also, we hope to develop this approach to the solution of three-dimensional inverse scattering problems in electromagnetic subsurface tomography.[15]

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Disclosure statement

No potential conflict of interest was reported by the authors.

References

[1] Tikhonov AN. Determination of the electrical properties of deep layers of the Earth's crust. Dokl. Acad. Nauk. SSSR. 1950;73:295–297.
[2] Cagniard L. Basic Theory of the Magneto-Telluric Method of Geophysical Prospecting. Geophysics. 1953;18:605–635.
[3] Wait JR. On the relation between telluric currents and the earth's magnetic field. Geophysics. 1954;19:281–289.
[4] Yungul SH. Magneto-telluric sounding three-layer interpretation curves. Geophysics. 1961;26:465–473.
[5] Marquardt DW. An algorithm for least-square estimation of non linear parameters. J. SIMA. 1963;11:431–441.
[6] Rodi W, Mackie RL. Nonlinear conjugate gradients algorithm for 2-D magnetotelluric inversion. Geophysics. 2001;66:174–187.
[7] Zhdanov MS, Tolstaya E. Minimum support nonlinear parametrization in the solution of a 3D magnetotelluric inverse problem. Inverse Prob. 2004;20:937–952.
[8] Zhdanov MS, Keller G. The geoelectrical methods in geophysical exploration. Amsterdam: Elsevier; 1994.
[9] Gaikovich KP. Inverse problems in physical diagnostics. New York (NY): Nova Science Publishers Inc.; 2004.
[10] Gelfand IM, Levitan BM. On the determination of a differential equation from its spectral function. Am. Math. Soc. Transl. 1955;2:253–304.
[11] Tikhonov AN. Solution of ill-posed problems. New York, NY: Winston; 1977.
[12] Barisheva MM, Gaikovich KP, Gaikovich PK, et al. Reflectometry sounding of inhomogeneities in periodic multilayer structures. In Proceedings of 12th International Conference on Transparent Optical Networks; Munich: ICTON-2010; 2010. Tu.P5.
[13] Gaikovich KP. Subsurface near-field scanning tomography. Phys. Rev. Lett. 2007;98:183902.
[14] Gaikovich KP, Gaikovich PK. Inverse problem of near-field scattering in multilayer media. Inverse Prob. 2010;26:125013.
[15] Gaikovich KP, Gaikovich PK, Maksimovitch YeS, et al. Pseudopulse near-field subsurface tomography. Phys. Rev. Lett. 2012;108:163902.
[16] Gaikovich PK, Sumin MI, Gaikovich KP. One-dimensional inverse scattering problem. In Proc. 13th International Conference on Transparent Optical Networks; Stockholm, Sweden: ICTON-2011; 2011. We.A2.4.
[17] Bakushinski AB, Kokurin MY. Iterative methods for approximate solution of inverse problems. Dordrecht: Springer; 2004.
[18] Kaltenbacher B, Neubauer A, Scherzer O. Iterative regularization methods for nonlinear problems. Berlin: de Gruyter; 2008.
[19] Scherzer O, Grasmair M, Grossauer H, et al. Variational methods in imaging. Vol. 167, Applied mathematical sciences. New York (NY): Springer; 2009.
[20] Bauer F, Hohage T, Munk A. Iteratively regularized Gauss–Newton method for nonlinear inverse problems with random noise. SIAM J. Numer. Anal. 2009;47:1827–1846.
[21] Bissantz N, Hohage T, Munk A. Consistency and rates of convergence of nonlinear Tikhonov regularization with random noise. Inverse Prob. 2004;20:1773–1789.
[22] Sumin MI. Regularized dual method for nonlinear mathematical programming. Comput. Math. Math. Phys. 2007;47:760–779.
[23] Sumin MI. Parametric dual regularization in a nonlinear mathematical programming. Vol. 11, Advances in mathematics research. New-York: Nova Science Publishers Inc.; 2010. Chapter 5; 103–134.

[24] Kanatov AV, Sumin MI. Sequential stable Kuhn–Tucker theorem in nonlinear programming. Comput. Math. Math. Phys. 2013;53:1078–1098.
[25] Sumin MI. Duality-based regularization in a linear convex mathematical programming problem. Comput. Math. Math. Phys. 2007;47:579–600.
[26] Uzawa H. Iterative methods for concave programming. In Studies in Linear and Nonlinear Programming. Stanford: Stanford Univ. Press; 1958. Chapter 10.
[27] Minoux M. Mathematical programming: theory and algorithms. New York, NY: Wiley; 1986.
[28] Ekeland I, Temam R. Convex analysis and variational problems. Amsterdam: North-Holland; 1976.
[29] Glowinski R, Lions JL, Trémoliéres R. Numerical analysis of variational inequalities. Amsterdam: North-Holland; 1981.
[30] Temam R. Navier–Stokes equations: theory and numerical analysis. Amsterdam: North-Holland; 1977.
[31] Hestenes MR. Multiplier and gradient methods. J. Optim. Theory Appl. 1969;4:303–320.
[32] Powell MJD. A method for nonlinear constraints in minimization problems. In: Fletcher R, editor. Optimization. New-York: Academic Press; 1969. p. 283–298.
[33] Bertsekas DP. Constrained optimization and Lagrange multiplier methods. New York (NY): Academic Press; 1982.
[34] Chan T, Tai X. Identification of discontinuous coefficients in elliptic problems using total variation regularization. SIAM J. Sci. Comput. 2003;25:881–904.
[35] De Cezaro A, Leitão A, Tai X-C. On piecewise constant level-set (PCLS) methods for the identification of discontinuous parameters in ill-posed problems. Inverse Prob. 2013;29:015003.
[36] Sumin MI. A regularized gradient dual method for the inverse problem of a final observation for a paraboic equation. Comput. Math. Math. Phys. 2004;44:1903–1921.
[37] Sumin MI. Parametric dual regularization in a linear-convex mathematical programming. In Computational optimization: new research developments. New-York: Nova Science Publishers Inc; 2010. Chapter 10; 265–311.
[38] Sumin MI. Stable sequential Kuhn-Tucker theorem in iterative form or a regularized Uzawa algorithm in a regular nonlinear programming problem. Comput. Math. Math. Phys. 2015;55:947–977.
[39] Warga J. Optimal control of differential and functional equations. Academic: New York (NY); 1972.
[40] Borwein JM, Strojwas HM. Proximal analysis and boundaries of closed sets in Banach space, Part I: Theory. Can. J. Math. 1986;38:431–452; Part II: Applications. Can. J. Math. 1987;39:428–472.
[41] Loewen PD. Optimal control via nonsmooth analysis. CRM Proceedings and Lecture Notes. V.2. Providence (RI): American Mathematical Society; 1993.
[42] Mordukhovich BS. Variational analysis and generalized differentiation, I: basic theory; II: applications. Berlin: Springer; 2006.
[43] Frick K, Grasmair M. Regularization of linear ill-posed problems by the augmented Lagrangian method and variational inequalities. Inverse Prob. 2012;28:104005.
[44] Frick K, Scherzer O. Regularization of ill-posed linear equations by the non- stationary augmented Lagrangian method. J. Integral Equ. Appl. 2010;22:217–257.
[45] De Loor JPJ. Dielectric properties of heterogeneous mixtures containing water. Microwave Power. 1968;3:67–73.
[46] Spiller E. Reflective multilayer coatings for the far UV region. Appl. Opt. 1976;15:2333–2338.
[47] Akhsakhalyan AD, Fraerman AA, Polushkin NI, et al. Determination of layered synthetic microstructure parameters. Thin Solid Films. 1991;203:317–326.
[48] Andreev SS, Barysheva MM, Chkhalo NI. multilayered mirrors based on la/b4c(b9c) for x-ray range near anomalous dispersion of boron ($\lambda \approx 6.7$ nm), Nucl. Instrum. Methods Phys. Res., Sect. A. 2009;603:80–82.
[49] Gaikovich KP, Gaikovich PK, Sumin MI. Inverse scattering problem in pseudopulse diagnostics of *periodic structures*. Proceedings of 4th International Conference on 2012 Mathematical Methods in Electromagnetic Theory; Kharkiv, Ukraine: MMET-2012; 2012, 390–393.