Ill-posed Inverse Problems Based on Volterra-Type Equations[1]
(invited)

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Abstract - As it is well known, inverse problems based on Volterra equations are, as a rule, well-posed. But in the case when a function should be retrieved in the range which is wider than the range where the right side of the equation is given, the solution appears an ill-posed inverse problem. A number of physical examples is given, and it is shown that such inverse problems could be successfully solved on the basis of Tikhonov's method of general discrepancy.

Introduction

Let us consider the Volterra-type equations of the 1-st and 2-nd kind:

$$\int_a^t K(t,s)\varphi(s)ds = f(t), \tag{1}$$

$$\varphi(t) + \lambda\int_a^t K(t,s)\varphi(s)ds = f(t). \tag{2}$$

These equations are practically well-posed in $[a,b]$, when the right side of (1) or (2) is given in the same range $a \le t \le b$. More exactly, the equation (2) has a continuous and unique solution, if the kernel and the right side of (1) are continuous in $[a,b]$. The equation (1) has the continuous solution, if there are continuous derivatives $\frac{df}{dt}$ and $\frac{\partial K}{\partial t}$, $f(a) = 0$, and $K(t,t) \ne 0$ in $[a,b]$.

There is the possibility of the new formulation of the problem for the Volterra-type equations. It appears, when the right side of equations (1) and (2) is given in the $[a,c]$, where $c < b$, i.e., when the retrieval range is wider than the range, in which the right side $f(t)$ is given. In that case (1) and (2) can be rewritten as

$$\int_a^c K(t,s)\varphi(s)ds = f(t) - \int_c^t K(t,s)\varphi(s)ds = F(t), \tag{3}$$

$$\lambda\int_a^c K(t,s)\varphi(s)ds = f(t) - \varphi(t) - \int_c^t K(t,s)\varphi(s)ds = F'(t). \tag{4}$$

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One can see that if we suppose that not only $f(t)$ is known in $[a,c]$ but also the function φ(s), we have the effective right sides $F(t)$ and $F'(t)$, and equations (3) and (4) are Fredholm integral equations of the 1-st kind relative the solution in $[c,b]$. Such equations are typical ill-posed problems. It is clear, that the solution of (1) and (2) in the whole range $[a,b]$ by $f(t)$ given in $[a,c]$ is still more complicated problem. In this case these integral equations are the ill-posed, type of which has yet no special name.

The most effective approach to solution of ill-posed integral equations is the Tikhonov's theory based on generalized discrepancy principle and the solution method of the same name [1].The main preference of Tikhonov's method consist in the uniform convergence of the retrieval error to zero at mean square convergence of right side errors. As it is in all ill-posed problems, its accuracy could be determined only on the basis of numerical simulation.

Physical problems based on ill-posed Volterra-type equations

Physical problems related with integral equations are, as a rule, inverse problems. Some of them consist in the solution of Volterra equations, and could be considered in the described above formulation as ill-posed problems. Some examples are presented here.

1. *Refraction inverse problem in a spherical symmetry medium* [2,3].

a. Limb-viewing geometry [2].

For limb-viewing measurements the refraction inverse problem can be expressed as the Volterra-type integral equation of the 1-st kind (the dependence of refraction ε on radial distance of ray perigee):

$$10^{-6}\int_{p_0}^{p_{\max}} \frac{dN}{dp}(p)\frac{-2p}{\sqrt{p^2-p_0^2}}\,dp = \varepsilon(p_0)\ ,\quad p_1 \le p_0 \le p_{\max}\ , \qquad (1)$$

where $p = nr$, $n_0 = n(r_0)$, $p_0 = n_0 r_0$, r, r_0 are radial distances, $N = 10^6(n\text{-}1)$ is refraction index, n is refractive index.

b. Immersion geometry [3].

The dependence of refraction on radial position (distance) of the source or receiver in the medium can be expressed as Volterra integral equation of the 2-nd kind:

$$N(p_0) - \int_{p_0}^{p_{\max}} N(p)\frac{pp_0\cos\theta(p_0)}{\left[p^2 - [p_0\cos\theta(p_0)]^2\right]^{3/2}}\,dp = 10^6\, tg\theta\ \varepsilon(p_0)\ ,\quad p_1 \le p_0 \le p_{\max}\ , \quad (2)$$

where θ is the elevation angle of the ray at the source position.

If one considers the equations (1) or (2) in the case, when their right side is given in the region $p_1 \le p_0 \le p_2$, $p_2 < p_{\max}$, the solution for the region $p_1 \le p \le p_{\max}$ becomes an ill-posed problem. Similar equations describe the radiometry inverse problems of limb-viewing and immersion remote sensing of planet atmospheres [4].

2. *Diagnostics of the superconductive films in a strong electromagnetic field* [5-6].

The measured dependence of averaged over the conductor surface resistance on magnetic field amplitude in the case of one-dimensional distribution of magnetic field H in a rectangular cavity resonator is related with the true resistance dependence $R_s(H)$ as

$$< R_s(H_m) > \; = \frac{4}{\pi H_m} \int_0^{H_m} \frac{(H/H_m)^2}{\sqrt{1-(H/H_m)^2}} \, R_s(H)\, dH \,, \qquad 0 \le H_m \le H_{max} \,, \qquad (3)$$

The inverse problem of $R_s(H)$ retrieval in the range $0 \le H_m \le H_{max}$ becomes ill-posed in real conditions, when the measurements region is limited at low magnetic field values, and there are measurements only in the range $H_2 \le H_m \le H_{max}$.

3. Thermal history inverse problems.

a. Thermal conductivity equation for half-space.

Let us consider the homogeneous half-space $z \le 0$ with the constant parameters: thermal diffusivity coefficient a^2. If we have boundary condition for temperature $T(0,t) = T_0(t)$, then the dynamics of the temperature distribution inside the half-space can be determined from thermal conductivity equation as a function of depth and time as follows:

$$T(z,t) = \int_{-\infty}^{t} T_0(t) \frac{-z}{\sqrt{4\pi a^2 (t-\tau)^3}} \exp\left(-\frac{z^2}{4a^2(t-\tau)}\right) d\tau \,. \qquad (4)$$

The inverse problem consist of retrieval of the boundary condition $T_0(t)$ by measurements $T(z,t)$. There are two possibilities: the first of them (Tikhonov's [1]) is based on measurements of depth profile $T(z)$ at time t_0, and the second (considered here as ill-posed Volterra-type equation) is based on measurements $T(t)$ at some arbitrary depth z_0 in the range $a \le t \le b$. The retrieval in this, second, case should be found in the region $[c,b]$, where $c < a$. For the solution the necessary condition is $T_0(t) \equiv 0$ at $t < c$ (otherwise, it will be unaccounted source of error).

b. Thermal conductivity equation for space with the spherically symmetric source.

If we have the homogeneous space $r \ge 0$ with the boundary condition $T(R,t) = T_0(t)$ on the sphere $r = R$, the temperature evolution in the region $r > R$ is determined by

$$T(r,t) = \int_{-\infty}^{t} T_0(t) \frac{R(r-R)}{r\sqrt{4\pi a^2 (t-\tau)^3}} \exp\left(-\frac{(r-R)^2}{4a^2(t-\tau)}\right) d\tau \qquad (5)$$

The ill-posed Volterra-type equation for (5) is the same as for (4) - to retrieve the $T_0(R,t)$ in the range $a \le t \le b$ by $T(r_0,t)$ at some arbitrary radial distance r_0 in the region $[c,b]$, $c < a$.

c. Retrieval of temperature evolution of media by thermal emission dynamics.

More sophisticated inverse problems are based on simultaneous solution of thermal conductivity and thermal emission transfer equations [7]. The brightness temperature of upward thermal radio emission of half-space $z \le 0$ at wavelength λ is determined from emission transfer equation, assuming that the reflection on half-space interface is absent :

$$T_b(\lambda) = \int_{-\infty}^{0} T(z)\gamma(\lambda)\exp(\gamma z) dz \quad , \qquad (6)$$

where γ(λ) is the absorption coefficient.

The substitution of (4) into (6) gives [7]:

$$T_b(t) = \int_{-\infty}^{t} T_0(t)[\frac{\gamma a}{\sqrt{\pi(t-\tau)}} - (\gamma a)^2 erfc(\gamma a\sqrt{t-\tau})e^{(\gamma a)^2(t-\tau)}]d\tau \qquad (7)$$

If the function $T_b(t)$ is known in the whole region (-∞,*b*], the equation (7) has the exact solution [7]:

$$T_0(t) = T_b(t) + \frac{1}{\gamma a}\int_{-\infty}^{t} (T_b(t) - T_b(\tau))\frac{d\tau}{\sqrt{\pi(t-\tau)^3}} \quad . \qquad (8)$$

Otherwise, if $T_b(t)$ is known in some limited region [a,b], the problem of retrieval of $T_0(t)$ in the region [*c*,*b*], where $c < a$, is also the Volterra-type ill-posed problem.

For the sphere case (see item b), there are different possibilities to choose the beam geometry, which determines the form of emission transfer integral. The most simple equation corresponds to the case of radial directed (from sphere) measurements:

$$T_b(t) = \int_{-\infty}^{t} T_0(\tau)[\int_{R}^{\infty} \frac{\gamma}{\sqrt{4\pi a^2}}\frac{R(r-R)}{r}e^{-\frac{(r-R)^2}{4a^2(t-\tau)}-\gamma(r-R)}dr]\frac{d\tau}{\sqrt{(t-\tau)^3}} \quad . \qquad (9)$$

More common case, when a ray perigee radial distance $r_0 \neq 0$ ($r_0 > R$), the radiobrightness can be expressed as

$$T_b(t) = \int_{-\infty}^{t} T_0(\tau)[\int_{r_0}^{R_0} \frac{\gamma}{\sqrt{4\pi a^2}}\frac{R(r-R)}{\sqrt{r^2-r_0^2}}e^{-\frac{(r-R)^2}{4a^2(t-\tau)}-\gamma(\sqrt{R_0^2-r_0^2}-\sqrt{r^2-r_0^2})}dr + $$
$$+\int_{r_0}^{\infty} \frac{\gamma}{\sqrt{4\pi a^2}}\frac{R(r-R)}{\sqrt{r^2-r_0^2}}e^{-\frac{(r-R)^2}{4a^2(t-\tau)}-\gamma(\sqrt{R_0^2-r_0^2}+\sqrt{r^2-r_0^2})}dr]\frac{d\tau}{\sqrt{(t-\tau)^3}} \quad , \qquad (10)$$

where R_0 is the radial distance of the receiver. The ill-posed Volterra-type equations for (9) and (10) are the same as for (7). For the equation (10) there is also the possibility to formulate the limb-viewing inverse problems, similar with refraction inverse problems (see equations (1) and (2)), using the dependence $T_b(r_0)$.

Let us consider the solution of equation (7) in detail as a typical example of ill-posed Volterra-type equations. If to introduce the time parameter $\Gamma=1/(\gamma a)^2$, which is a typical time of the heating of the medium at the skin-depth $z_s=1/\gamma$, it is possible to rewrite (7) in simpler, dimensionless form, using dimensionless parameters $r=t/\Gamma$, $\rho=\tau/\Gamma$:

$$T_b(r) = \int_{-\infty}^{r} T_0(\rho)[\frac{1}{\sqrt{\pi(r-\rho)}} - erfc(\sqrt{r-\rho})e^{(r-\rho)}]d\rho \qquad (11)$$

To solve such a problem it is necessary to use additional (*a priori*) information about the exact solution. This information determines a regularization method. There are various approaches, but in the present paper Tikhonov's method of generalized discrepancy is applied, which uses the common information about the exact solution as a function [1]. It is supposed in this method that the exact solution belongs to the set of square-integrable functions with square-integrable derivatives. The results of numerical simulation give us the retrieval accuracy at various levels of the radiobrightness error. It appears possible to retrieve the function $T_0(\rho)$ in the range $[c/\Gamma, b/\Gamma]$ by measurements $T_b(r)$ in the range $[a/\Gamma, b/\Gamma]$, $c < a$, up to values $a - c \approx 2 \div 5\ \Gamma$ at measurement accuracy about 1%. The main preference of Tikhonov's method consist of the uniform convergence of the retrieval error to zero at mean square convergence of measurement errors. As in all ill-posed problems, this convergence is slower than it is in well-posed problems.

The numerical algorithm of the Tickonov's method (the same as in [6]) was applied to the retrieval of diurnal temperature dynamics of soil by its thermal radio emission evolution measurements [8]. The measurements have been carried out using radiometers at wavelengths 0.8; 3; 9, and 13 cm under metallic screen (to eliminate the influence of reflection on interface air-soil). In the Fig.1 is shown an example of retrieval of the surface temperature in time interval from 15^h (r = 0) to 12^h20^m (r = 8.25) next day by measurements of radiobrightness at wavelength 3 cm in time interval from 3^h10^m (after midnight) to 12^h20^m . The parameters values were: $a^2 = 0.001$ cm^2/s, $\gamma = 0.33$ cm^{-1}, $\Gamma = 2.55\ ^h$. So, $a = 15^h$, $b = 12^h20^m$, $c=3^h10^m$.

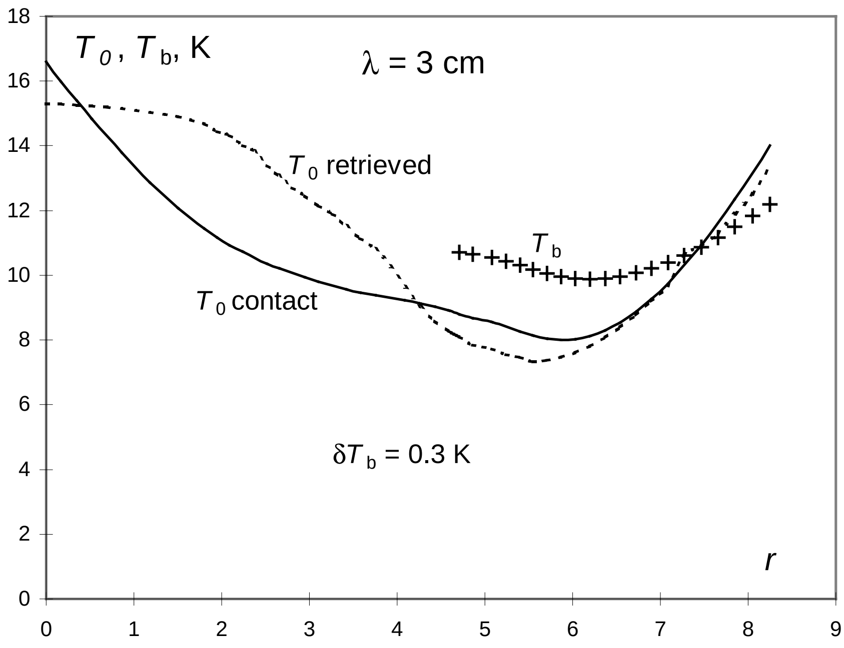

Fig.1.

It is possible to see that the retrieval in the time interval t > a, where there are measurements $T_b(t)$, is very close to contact measured dynamics $T_0(t)$. At $c \leq t \leq a$ the accuracy of the surface temperature history retrieval reduces, but it appeared possible to retrieve the process of night surface cooling. It is clear that the problem is more difficult for retrieval of the thermal history than for retrieval of simultaneous surface temperature dynamics.

The retrieval of the surface temperature dynamics permit then to retrieve the temperature profile dynamics in the medium from the equation (4), and to calculate the thermal flux evolution [7].

Conclusions

The results of the solution of various physical problems based on Volterra-type integral equation in considered here ill-posed formulation show that the domain of definition of the solution consist of two very different sub-ranges. The first sub-region (which could be called «inner») coincides with the domain of definition of equations right side. The second (outer) sub-range is located outside the domain of definition of equations right side. The approximate solution in the outer region (as, for example, for the considered here in detail thermal history inverse problem) diverges to the exact one much more slowly than in the inner sub-region. In the inner sub-region the requirements to data accuracy could be very different in different physical problems, but always they are less than for outer sub-region. Moreover, in the outer sub-region the retrieval accuracy reduces with the distance to the boundary of inner sub-region. Considered here new formulation solves the problem of influence of unknown non-zero initial conditions on the solution of Volterra equations. No doubt, there are many possible applications of this approach, which remain unmentioned in this communication.

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