DUAL REGULARIZATION IN ONE-DIMENSIONAL INVERSE SCATTERING PROBLEM

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Abstract

The considered inverse problem of electromagnetic scattering is widely applied in the subsurface profiling of media permittivity. In previous works, mainly the non-linear integral equation for the scattered field has been in use. It has been solved in the Born approximation or, sometimes, iteratively – beyond this approximation. However, the solution of this ill-posed problem at each step of iterations faced difficulties. To overcome these difficulties, we propose to use the new approach based on the Lagrange formalism applied to initial differential equations (Maxwell's equations). That gives a possibility to obtain the solution of one-dimensional inverse problems of scattering beyond the range of applicability of the perturbation theory. Based on the developed theory, the solution algorithm has been worked out and applied to the simplest one-dimensional problem of low frequency geomagnetic profiling of conductivity of the earth crust.

Keywords: Dual regularization, Maxwell's equations, inverse scattering problem, electromagnetic sounding, earth crust

1. INTRODUCTION

The problem of one-dimensional electromagnetic geomagnetic sounding has been formulated firstly by A.N.Tikhonov [1] for the ultra low-frequency sounding of earth crust and solved by him for a discrete multilayered distribution of media conductivity. The frequency dependence of the effective depth of the received signal formation (skin-depth) of measured fields was in use in this method, applied further in the magnetotelluric exploration. The depth of sounding achieves several kilometers at lowest frequencies.

For the case of a continuous conductivity profile, in frameworks of the one-dimensional electromagnetic perturbation theory, this problem has been reduced to the solution of the non-linear integral equation of the 1-st kind that has been solved iteratively using Tikhonov's method of generalized discrepancy in [2,3]. Here we develop the dual-regularization approach [4].

2. THEORY

2.1. ONE-DIMENSIONAL INVERSE PROBLEM OF SCATTERING

If the distribution of a probing electric field in the non-perturbed medium with the permittivity ε_0 is $\mathbf{E_0}(\mathbf{r})$, the total field $\mathbf{E}(\mathbf{r})$ for the same medium with inhomogeneities $\varepsilon_1(\mathbf{r}')$ can be expressed as a sum of probing and scattered fields and obtained iteratively from the Fredholm equation of the 2-nd kind [5,6]:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E_0}(\mathbf{r}) + \mathbf{E}_1(\mathbf{r}) = \\ = \mathbf{E_0}(\mathbf{r}) - \frac{i\omega}{4\pi}\int_V \overleftrightarrow{\mathbf{G}}(\mathbf{r}',\mathbf{r})\varepsilon_1(\mathbf{r}')\mathbf{E}(\mathbf{r}')d\mathbf{r}', \quad (1)$$

beginning with the Born approximation

$$\mathbf{E}_1(\mathbf{r}) = -\frac{i\omega}{4\pi}\int_V \overleftrightarrow{\mathbf{G}}(\mathbf{r}',\mathbf{r})\varepsilon_1(\mathbf{r}')\mathbf{E}_0(\mathbf{r}')d\mathbf{r}', \quad (2)$$

where ω is the cyclic frequency. The Green tensor $\overleftrightarrow{\mathbf{G}}$ for homogeneous or multilayer media can be obtained using the input impedance formalism [5,6]. Equations (1,2) can also be used to solve the inverse scattering problem [6]. In the case of one-dimensional media, when plane waves are used as the probing field, the total field is expressed as $\mathbf{E}(\mathbf{r}) = \mathbf{E}(z)\exp(ik_x x + ik_y y)$ and the problem is much simplified and reduced to the equation

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$$\mathbf{E}_1(z) = -\frac{i\omega}{4\pi}\int_{z'} \varepsilon_1(z')\vec{\mathbf{g}}(z',z)\mathbf{E}_0(z')dz'. \quad (3)$$

In the ultra low frequency band, the analysis can be simplified further. The approximation of Leontovich's boundary conditions is valid in this band, so the field in the medium can be considered as a plane wave with components E_x, H_y of electric and magnetic field respectively that propagates in the nadir direction relative the earth surface. Also, the permittivity at low frequencies is determined by the conductivity σ as $\varepsilon = \varepsilon' - i\varepsilon'' \approx -4\pi i\sigma/\omega$. Maxwell's equations for the complex amplitudes of electric and magnetic field are written as

$$\frac{d^2E}{dz^2} - i\frac{4\pi\sigma(z)\omega}{c^2}E = 0, \quad H = i\frac{c}{\omega}\frac{dE}{dz}, \quad (4)$$

Fields are measured at the surface level in dependence on frequency:

$$E(z=0,\omega) = E_0(\omega),\ H(z=0,\omega) = H_0(\omega). \quad (5)$$

These data can be compared to frequency dependences for the homogeneous medium with the conductivity $\sigma(z) = \sigma(0) = \sigma_0$:

$$E^0(\omega,z) = E_0^0(\omega)\exp(\frac{z}{\delta} + i\frac{z}{\delta}), \quad (6)$$

$$H^0(\omega,z) = \frac{c(i-1)}{\omega\delta}E^0 = H_0^0(\omega)\exp(\frac{z}{\delta} + i\frac{z}{\delta}),$$

where $\delta = c/2\pi\omega\sigma_0$ is the skin-depth. Using differences $\Delta H = H_0 - H^0$, $\Delta E = E_0 - E^0$ of these fields in the first guess at the solution of the non-linear integral equation (3) or of the corresponding equation for the magnetic field, it is possible to solve the (3) iteratively, beginning with the Born approximation of perturbation theory. Results of the numerical study [3] have demonstrated serious limitations of such approach for large perturbations (typical in geology structures), when the Born approximation (first guess of iterative method) is inapplicable. To overcome these restrictions of perturbation theory, we develop here the new method, based on the theory [4,7], applied to initial Maxwell equations in its simplest version.

2.2. Method of dual regularization

Let us introduce new variables:

$$\mathrm{Re}\,E = x_1, \frac{d\,\mathrm{Re}\,E}{dz} = x_2, \mathrm{Im}\,E = x_3, \frac{d\,\mathrm{Im}\,E}{dz} = x_4.$$

Then the inverse problem for (4) in the range $z_0 \le z < 0$ lead to the equivalent problem of minimization

(7)

$$I_0(\xi,\sigma) \equiv \|\sigma\|^2 + |\xi|^2 \to \min,\ I_1(\xi,\sigma) = x_0(\omega),$$

$$\sigma \in D \equiv \{\sigma \in L_2(z_0,0) : 0 \le \sigma(z) \le \sigma_0\} \subset L_2(z_0,0)$$

$$\frac{dx}{dz} = A(\sigma(z)\omega)x, \quad x(z_0) = \xi,$$

$$I_1(\xi,\sigma)(\omega) \equiv x[\xi,\sigma](0) \equiv x[\xi,\sigma\omega](0) \equiv x_0(\omega),$$

$$z_0 < 0,\ \omega \in [\omega_1,\omega_2],\ \xi \in R^4,$$

$$A(\sigma(z)\omega) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -a\sigma(z)\omega & 0 \\ 0 & 0 & 0 & 1 \\ a\sigma(z)\omega & 0 & 0 & 0 \end{pmatrix},$$

$$x_0(\omega) = \begin{pmatrix} 1 \\ H_{02}(\omega)\omega/c \\ 0 \\ H_{01}(\omega)\omega/c \end{pmatrix},$$

where $a = 4\pi/c^2$. It is possible to prove that this problem has a solution (there is no uniqueness). Denote the value of minimizing functional I_0 at this solution as I_0^*. In the considered nonlinear problem it is necessary to use the modified Lagrange function in the dual regularization of (7) [7]. It can be build as

$$\begin{aligned} &L_\mu(\xi,\sigma,\lambda) \equiv \|\sigma\|^2 + |\xi|^2 + \\ &+\int_{\omega_1}^{\omega_2} (\lambda(\omega),\ x[\xi,\sigma](0,\omega) - x_0(\omega))d\omega + \\ &+\mu(\sqrt{\int_{\omega_1}^{\omega_2} |x[\xi,\sigma](0,\omega) - x_0(\omega)|^2\, d\omega} + \\ &+\int_{\omega_1}^{\omega_2} |x[\xi,\sigma](0,\omega) - x_0(\omega)|^2\, d\omega) \equiv \\ &\equiv \|\sigma\|^2 + |\xi|^2 + (\lambda, I_1(\xi,\sigma) - x_0(\omega)) + \\ &+\mu(\|I_1(\xi,\sigma) - x_0(\omega)\| + \|I_1(\xi,\sigma) - x_0(\omega)\|^2). \end{aligned} \quad (8)$$

The dual problem (Tikhonov regularization with the regularization parameter α) of maximizing the concave functional on the Hilbert space $L_2^4(\omega_1,\omega_2)$:

$$V_\mu(\lambda) - \alpha\|\lambda\|^2 \to \max,\ \|\lambda\| \le \mu, \quad (9)$$

$$V_\mu(\lambda) \equiv \min_{\sigma\in D} L_\mu(\xi,\sigma,\lambda), \quad \lambda \in L_2^4(\omega_1,\omega_2),\ \mu > 0.$$

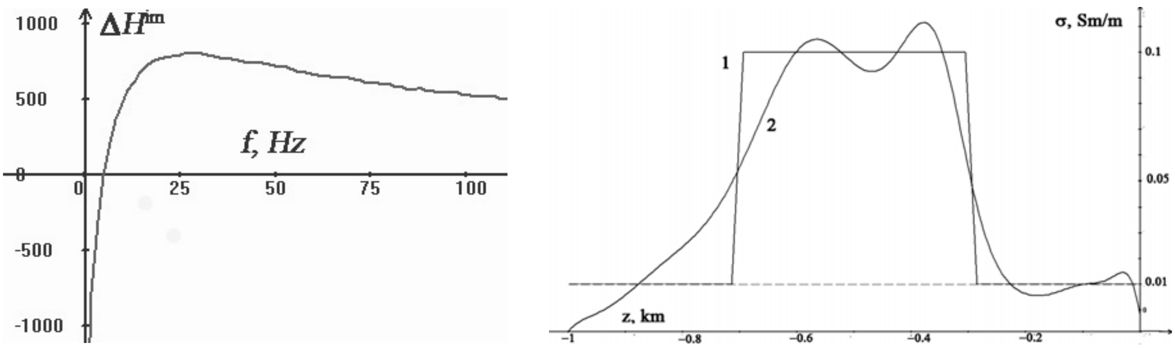

Fig. 1. Numerical modeling of the retrieval of the conductivity profile. Left, "measured data" (Im (ΔH)) in arbitrary units versus frequency $f = \omega/2\pi$ at the *rms* of the random error δH_0 = 1%; right, 1 – initial profile, 2 – retrieval results.

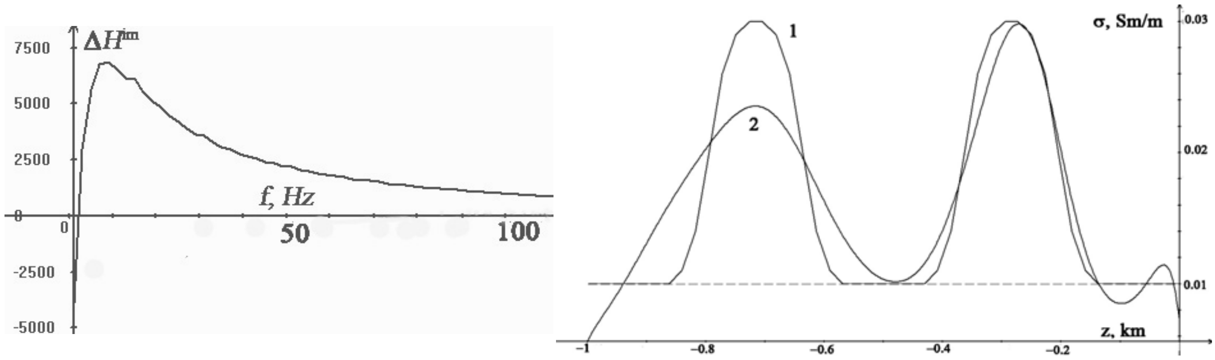

Fig. 2. The same, as in Fig. 1, but for the conductivity profile with two maxima.

Details of the dual regularization algorithm are described in [4,7]. In this paper we present results of this problem solution based on a simplified approach – the minimization of the discrepancy functional $\| I_1(\sigma,\xi) - x_0(\omega) \|^2$ in (8) under the stipulation that $x \to 0$, when $z \to -\infty$.

3. Numerical simulation

In Figs. 1, 2 one results of the numerical simulation of the retrieval of conductivity profiles are shown. One can see in Fig. 1 a good retrieval of a very sharp inhomogeneity (layer with enhanced conductivity). The second example in Fig. 2 demonstrated the possibility of the developing approach to retrieve more complicated profiles of conductivity. Advantages of this approach as compared with the solution of integral equation [3] are quite obvious.

4. Permission to Publish

The authors are responsible for all material contained in the manuscript they submit.

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