USE OF THE HEAT-EVOLUTION EQUATION TO DETERMINE TEMPERATURE DISTRIBUTION IN HALF-SPACE BY THERMAL RADIO EMISSION

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An analytic solution is obtained for the problem of determination of the temperature profile *of a half-space according to the evolution of the brightness temperature of its thermal radio emission by joint solution of the equations of radiation transport and thermal conductivity.*

1. Introduction. Remote determination of the temperature distribution of a medium according to thermal radio emission is employed to solve problems of geophysics, radio astronomy, biomedical diagnosis, and atmospheric sensing. The range of problems of subsurface sensing is ever-widening. The greatest difference from atmospheric problems, which also causes the greatest difficulties, is the presence of reflection and scattering by interfaces in subsurface sensing. The variations of the brightness temperatures of a medium caused by the spectral and temperature dependences of the reflection coefficient, often exceed the effect associated with profound temperature inhomogeneity.

For just this reason, perhaps, problems of subsurface sensing first began to be solved in radio astronomy, where, for example, the variations of the moon's brightness temperature are 150-200 K and greatly exceed the variations due to uncertainty of the reflection coefficient (≤10 K) [1]. This problem received further development in medical applications for radiometric determination of internal body temperature, where a contact method was employed that used compensating noise [2, 3| and the problem of the formation of the thermal radiation of a multi-layer medium was solved [4]. Л great deal of interest in the possibilities (if subsurface sensing in geophysics has been shown [5]. Appreciable practical progress has been achieved in this area |d. 8], in which the conflicting factors of reflection and scattering have been eliminated by means of a special measurement procedure based on radiation reception by an antenna system situated below a flat metal shield. This procedure has been used to solve problems of sensing of the thermally stratified surface layer of an aqueous medium [6| and ground temperature profiles under summer and winter conditions [7, 8].

Solution of the enumerated problems comes down to determination of the temperature profile $T(z)$ of a half-space $z \leq 0$ from the brightness temperatures T_b of the thermal radio emission

$$T_b(\lambda) = (1 - R) \int_{-\infty}^{0} T(z)\gamma(\lambda)\exp[\gamma(\lambda)z]dz, \tag{1}$$

where γ is the absorption coefficient of the medium, λ is wavelength, and R is the reflection coefficient (below, we shall assume $R = 0$). It is difficult to solve (1) because this Fredholm equation of the first kind is incorrect and its solution is impossible without additional data on $T(z)$ [9]. In a specific physical situation, these data can have a particular form that determines the method of regularization of (1). Statistical data on the average temperature profile and its covariant interlevel relations (the so-called "statistical regularization" method) [10| and repression methods |11] arc usually employed in thermal sensing of the atmosphere. Sometimes, the form of the function $T(z)$ is known - for example, in sensing of the structure of internal waves in the atmosphere [12] or in measurements of frozen ground (8]. Then, the problem is reduced to solution of a system of equations

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in the parameters that determine that function. In other cases, it is known that $T(z)$ belongs lo a compact class of functions (monotone, convex, or bounded). Then, a solution is obtained by minimization of the residual by gradient methods [13]. Highly effective are ihe methods of Tikhonov el al. (9), which use the quadratic summability or existence of derivatives of 7"(z) and are based on minimization of the regularizing functional (method of generalized residual [9]). This method has been applied successfully earlier |6-8].

New possibilities of remote sensing are opened if we use the fact that the subsurface temperature distribution is not arbitrary but satisfies the equation of thermal conductivity for definite boundary conditions. This allows us lo examine and use the time dependence of the brightness temperature. U becomes possible lo obtain an analytic solution of Eq. (1).

2. Heat-Evolution Equations for Brightness Temperature. The idea of using the thermal-conductivity equation to solve (1) arose in the first work on subsurface sensing of the moon, where a known solution of the thermal-conductivity equation for a periodic boundary condition was used in (1) |1|. A joint solution of the equations of radiation transport and thermal conductivity has been obtained [7, 14] for arbitrary boundary conditions in the form of integral relations between the observed brightness temperatures and the preceding evolution of surface temperature (or heal flux). These relations are called heat-evolution equations.

When the boundary condition for the temperature $T(z,t)$ of a half-space $z \le 0$ (t is time) with coefficient of thermal diffusivity a^2 has the form

$$T(0,t) = T_0(t), \tag{2}$$

the first heat-evolution equation is valid for the brightness temperature

$$T_b(t) = \int_{-\infty}^{t} T_0(\tau)\left[\frac{\gamma a}{\sqrt{\pi(t-\tau)}} - (\gamma a)^2 \operatorname{erfc}(\gamma a\sqrt{t-\tau})\exp[(\gamma a)^2(t-\tau)]\right]d\tau.$$

If the boundary condition for the heal flux J_0 through the surface $z = 0$ is given in the form

$$\frac{dT}{dz}(0,t) = -\frac{1}{k}J_0(t),$$

the second heat-evolution equation is obtained:

$$T_b(t) = -\int_{-\infty}^{t} J_0(\tau)\,\frac{a^2\gamma}{k}\operatorname{erfc}(\gamma a\sqrt{t-\tau})\exp[(\gamma a)^2(t-\tau)]\,d\tau,$$

where k is the coefficient of thermal conductivity.

Equations (3) and (5) have been used [7, 14] to solve a number of problems of remote determination of the temperature dynamics and heat exchange of a surface with the atmosphere. In particular, Eqs. (3) and (5) were solved numerically as Volterra linear integral equations of the first kind (with a variable upper limit) for $T_0(\tau)$ or $J_0(\tau)$. Then, the known solutions of the thermal-conductivity equation were used 10 determine the dynamics of the subsurface profile $T(z,t)$. Ii is important lo note that a reconstruction was accomplished from observations of T_b only at one wavelength.

However, the numerical solution of integral Eqs. (3) and (5) is fairly complicated, and limits on dimensionality are inevitable in calculations by personal computers of moderate power. This circumstance and the desire to obtain results at greater depths stimulated the search for an analytic solution of (3) and (5) that would allow direct calculation of the profile $T(z,t)$ from specified dynamics $T_b(t)$. The derivation of this result is (he principal pan of the work.

3. Solution of Heat-Evolution Equations. We shall start with Eq. (5). We write it in abbreviated form

$$T_b(t) = -\int_{-\infty}^{t} J_0(\tau)K(t,\tau)\,d\tau \tag{6}$$

and differentiate it with respect to t:

$$\frac{dT_b}{dt} = -\frac{a^2\gamma}{k}J_0(t) - \frac{a^2\gamma}{k}\int_{-\infty}^{t} J_0(t)(\gamma a)^2\left[\frac{k}{a^2\gamma}K(t,\tau) - \frac{1}{\sqrt{\pi}\gamma a\sqrt{t-\tau}}\right]d\tau. \tag{7}$$

Noting that the second term in (7) coincides with (6) with accuracy to a factor, we have

$$\frac{dT_b}{dt} = (\gamma a)^2 T_b - \frac{a^2\gamma}{k} J_0(t) + \frac{a^2\gamma}{k}\frac{(\gamma a)}{\sqrt{\pi}} \int_{-\infty}^{t} J_0(\tau)\frac{d\tau}{\sqrt{t-\tau}} . \tag{8}$$

If we let $\chi = \gamma a\sqrt{\pi}$, $f(t) = -(k/\gamma a^2)(dT_b/dt - (\gamma a)^2 T_b)$ we can rewrite (8) as

$$J_0(t) = \chi \int_{-\infty}^{t} J_0(\tau)\frac{d\tau}{\sqrt{t-\tau}} + f(t) \tag{9}$$

which is a Volterra equation of the second kind with Abel's kernel. We shall use an iterated-kernel method [15| to solve it. After convolution of the left and right sides of Eq. (9) with the kernel of the integral in (9), we have

$$\chi \int_{-\infty}^{t} J_0(\tau)\frac{d\tau}{\sqrt{t-\tau}} = \chi^2\pi \int_{-\infty}^{t} J_0(\tau)d\tau + \chi^2 \int_{-\infty}^{t} f(t)\frac{d\tau}{\sqrt{t-\tau}} . \tag{10}$$

On the strength of Eq. (9),

$$\chi \int_{-\infty}^{t} J_0(\tau)\frac{d\tau}{\sqrt{t-\tau}} = J_0(t) - f(t),$$

so that (10) takes the form

$$J_0(t) = (\gamma a)^2 \int_{-\infty}^{t} J_0(\tau)d\tau + f(t) + \frac{\gamma a}{\sqrt{\pi}} \int_{-\infty}^{t} f(\tau)\frac{d\tau}{\sqrt{t-\tau}}. \tag{12}$$

Differentiating (12) with respect to time, we have

$$\frac{dJ_0}{dt} = (\gamma a)^2 J_0 + f'(t) + \frac{\gamma a}{\sqrt{\pi}} \int_{-\infty}^{t} f'(\tau)\frac{d\tau}{\sqrt{t-\tau}} . \tag{13}$$

Equation (13) is easily integrated:

$$J_0(t) = \int_{-\infty}^{t} f'(\tau)\exp[(\gamma a)^2(t-\tau)]\mathrm{erfc}(-\gamma a\sqrt{t-\tau})d\tau. \tag{14}$$

Then, we substitute into (14) the expression $f' = -(k/(a^2\gamma)(T_b'' - (\gamma a)^2 T_b')$,

$$J_0(t) = \frac{k}{a^2\gamma}\int_{-\infty}^{t} (\gamma a)^2 T_b'(\tau)\exp[(\gamma a)^2(t-\tau)]\mathrm{erfc}(-\gamma a\sqrt{t-\tau})d\tau - \\ - \frac{k}{a^2\gamma}\int_{-\infty}^{t} T_b''(\tau)\exp[(\gamma a)^2(t-\tau)]\mathrm{erfc}(-\gamma a\sqrt{t-\tau})d\tau \tag{15}$$

and integrate the second term in (15) by parts

$$J_0(t) = \frac{k}{a^2\gamma}\Big[\int_{-\infty}^{t} (\gamma a)^2 T_b'(\tau)\exp[(\gamma a)^2(t-\tau)]\mathrm{erfc}(-\gamma a\sqrt{t-\tau})d\tau - \\ - T_b'(\tau)\exp[(\gamma a)^2(t-\tau)]\mathrm{erfc}(-\gamma a\sqrt{t-\tau})\Big|_{-\infty}^{t} - \tag{16}$$

$$- \int_{-\infty}^{t} (\gamma a)^2 T_b'(\tau) \left(\frac{1}{\sqrt{\pi}\gamma a \sqrt{t-\tau}} + \exp[(\gamma a)^2 (t-\tau)] \operatorname{erfc}(-\gamma a\sqrt{t-\tau}) d\tau \right) \Bigg] .$$

As a result, we obtain an inversion formula for Eq. (5):

$$J_0(t) = - \frac{k}{a^2 \gamma} \left(T_b'(t) + \gamma a \int_{-\infty}^{t} T_b'(\tau) \frac{d\tau}{\sqrt{\pi(t-\tau)}} \right). \tag{17}$$

The solution of Eq. (3) is obtained from a known formula that links $J_Q(t)$ and $T_0(t)$ [16]:

$$T_0(t) = - \frac{a}{k} \int_{-\infty}^{t} J_0(t) \frac{d\tau}{\sqrt{\pi(t-\tau)}}. \tag{18}$$

Substituting (17) into (18), we have

$$T_0(t) = T_z(t) + \frac{1}{\gamma a} \int_{-\infty}^{t} T_b'(\tau) \frac{d\tau}{\sqrt{\pi(t-\tau)}}. \tag{19}$$

Formulas (17) and (19) allow the evolution of the temperature profile $T(z,t)$ to be determined by solution of the thermal-conductivity equation. Thus, substituting (19) into the known formula [16],

$$T(z,t) = -\int_{-\infty}^{t} T_0(\tau) \frac{z}{\sqrt{4\pi a^2 (t-\tau)^3}} \exp\left(- \frac{z^2}{4a^2(t-\tau)} \right) d\tau, \tag{20}$$

we have

$$T(z,t) = - \int_{-\infty}^{t} T_b(\tau) \frac{z}{\sqrt{4\pi a^2 (t-\tau)^3}} \exp\left(- \frac{z^2}{4a^2(t-\tau)} \right) d\tau -$$
$$- \int_{-\infty}^{t} \left(\frac{1}{\gamma a} \int_{-\infty}^{\tau} T_b'(s) \frac{ds}{\sqrt{\pi(\tau-s)}} \right) \frac{z}{\sqrt{4\pi a^2 (t-\tau)^3}} \exp\left(- \frac{z^2}{4a^2(t-\tau)} \right) d\tau. \tag{21}$$

Changing the order of integration in the second term of (21) and making performing the necessary transformations, we have

$$T(z,t) = - \int_{-\infty}^{t} T_b(\tau) \frac{z}{\sqrt{4\pi a^2 (t-\tau)^3}} \exp\left(- \frac{z^2}{4a^2(t-\tau)} \right) d\tau +$$
$$+ \frac{1}{\gamma a} \int_{-\infty}^{t} T_b'(\tau) \exp\left(- \frac{z^2}{4a^2(t-\tau)} \right) \frac{d\tau}{\sqrt{\pi(t-\tau)}}. \tag{22}$$

And, finally, integrating the second term of (22) by parts, we obtain the desired formula:

$$T(z,t) = \int_{-\infty}^{t} T_b(\tau) \exp\left(- \frac{z^2}{4a^2(t-\tau)} \right) \left[-z - \frac{1}{\gamma} \left(1 - \frac{z^2}{2a^2(t-\tau)} \right) \right] \frac{d\tau}{\sqrt{4\pi a^2 (t-\tau)^3}}. \tag{23}$$

Thus, the temperature profile is represented as an integral of the brightness-temperature evolution, i.e., the problem of single-wave temperature sensing has an exact solution and is correct (which is indicated by the results of numerical solution of Eqs. (3) and (5) [14]). It should be noted that the maximum sensing depth when (23) is used is not limited by the skin-layer thickness, as in the solution of (1) without allowance for the thermal-conductivity equation.

Let us consider the asymptotic behavior of the obtained equations with an unbounded rise in the absorption coefficient of the medium ($\gamma \to \infty$), when $T_b \approx T_0$ follows from (1) (assuming $R = 0$). Equation (23) becomes formula (20) in this case, Eq. (19) yields the natural result $T_0(t) = T_b(t)$ and Eq. (17) is transformed to the formula

$$J_0(t) = -\frac{k}{a} \int_{-\infty}^{t} \frac{dT_0}{d\tau}(\tau) \frac{d\tau}{\sqrt{\pi(t-\tau)}}. \quad (24)$$

which is the well-known relationship between heat-flux evolution and surface temperature (see Landau and Lifshits [16|, for example).

The obtained equations arc very compact and have a clear physical meaning.

The results present new possibilities for radiometric sensing. Single-wave monitoring of the temperature profile of a medium, surface temperature, and heal flux through the boundary of a half-space is realized on the basis of relations (17), (19), and (24). The wider possibilities of these equations, as compared with numerical integration of Eqs. (3) and (5), are entirely obvious.

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