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ИЗВЕСТИЯ ВУЗ. РАДИОФИЗИКА

(IZVESTIYA VUZ. RADIOFIZIKA)

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STOCHASTIC APPROACH TO THE RESULTS OF SIMULTANEOUS SOLUTION OF EMISSION TRANSFER AND THERMAL CONDUCTIVITY EQUATIONS

K. P. Gaikovich UDC 621.371:526.2+551.526+528.811+551.501

A stochastic theory of the temperature distribution and thermal radio emission of a medium (half-space) is developed on the basis of the results of simultaneous solution of thermal emission transfer and thermal conductivity equations. Expressions for the covariance functions of the temperature profile and brightness temperature as functions of the statistical parameters of the half-space surface temperature, which is considered a random, function of time, are found. Estimates of a temperature regression by thermal emission are analyzed using the expressions obtained.

1. INTRODUCTION

The authors [1-6] have developed a theory of thermal radio emission of a medium (half-space) whose temperature distribution depends on the dynamics of boundary conditions such as surface temperature or thermal flow through the surface of the medium. Using the simultaneous solution of thermal emission transfer and thermal conductivity equations, we found expressions for the brightness temperature of the medium in the form of a time integral of those boundary conditions [1-3]. Thereafter we transformed those equations [4-6] to express the boundary conditions and temperature profile of the medium through the evolution of its brightness temperature. Thus, we found a correct solution to the problem of one-wave radiometric remote sensing of the temperature profile of a medium.

We used those results in [4-6] for radiometric study of the diurnal thermal dynamics of soil (we used measurements of the brightness temperature dynamics of the thermal radio emission of soil at wavelengths of 0.8 and 3 cm) and of the atmospheric boundary layer (we used measurements of the intrinsic thermal radio emission of the atmosphere at a wavelength of 0.5 cm at the center of the oxygen absorption band).

However, long-term measurements of the dynamics of thermal emission are not always possible or convenient for control of the temperature profile of the medium. Also, there are methods of temperature profile reconstruction from the spectrum or angular dependence of brightness temperatures which are measured at an arbitrary instant of time. These methods involve the solution of Fredholm's ill-posed integral equation of first kind for brightness temperature. This equation cannot be solved without using a priori information about the properties of the desired function, including information about the smoothness, differentiability, and membership of $T(z)$ in a compact class (A. N. Tikhonov's method), or statistical information [12-14]. Statistical methods involve the use of the covariance functions and statistical parameters of the thermal radio emission and temperature profile. Empirical statistical characteristics, which are determined by measurements, are used in practice. For example, sets of weather-balloon data received at meteorological sounding stations are used to reconstruct the temperature profile of the atmosphere [12-14]. A. N. Tikhonov's methods are employed to solve radio thermometry problems for the boundary layer of the atmosphere and soil, since it is difficult to obtain the necessary data in this case [7-11]. However, the results of simultaneous solution of emission transfer and thermal conductivity equations can also be used for theoretical determination of the necessary correlation functions if the boundary condition for temperature u considered a random function of time.

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2. STATEMENT OF THE PROBLEM

Consider first a uniform half-space $z \le 0$ with constant parameters — the thermal conductivity coefficient a^2 and the thermal radio emission absorption coefficient γ. If a boundary condition is assigned for temperature $T(0,t) = T_0(t)$, then the temperature distribution of the half-space as a function of depth and time is determined from the thermal conductivity equation

$$T(z,t) = \int_{-\infty}^{t} T_0(\tau) \frac{-z}{\sqrt{4\pi a^2 (t-\tau)^3}} \exp\left(-\frac{z^2}{4a^2(t-\tau)}\right) \mathrm{d}\tau . \tag{1}$$

The brightness temperature of the ascending thermal emission at wavelength λ is found from the emission transfer equation

$$T_\mathrm{b}(t) = \int_{-\infty}^{0} T(z)\gamma(\lambda) e^{\gamma(\lambda)z}\,\mathrm{d}z , \tag{2}$$

where it is assumed for simplicity that the reflectivity of the half-space is equal to zero.

The simultaneous solution of Eqs. (1) and (2) enables one to express the brightness temperature through the boundary condition

$$T_\mathrm{b}(t) = \int_{-\infty}^{t} T_0(\tau) \left[\frac{\gamma a}{\sqrt{\pi(t-\tau)}} - (\gamma a)^2 \operatorname{erfc}\left(\gamma a \sqrt{t-\tau}\right) e^{(\gamma a)^2 (t-\tau)} \right] \mathrm{d}\tau . \tag{3}$$

A similar expression was obtained for an inhomogeneous medium [6] by using the properties of a Duhamel integral

$$T_\mathrm{b}(t) = \int_{-\infty}^{t} T_0(\tau) \frac{\partial}{\partial t} T_\mathrm{b}^{(1)}(t-\tau)\,\mathrm{d}\tau , \tag{4}$$

where $T_\mathrm{b}^{(1)}(t\text{-}\tau)$ is the response of the brightness temperature to a single jump of the surface temperature (the boundary condition is a Heaviside function):

$$T^{(1)}(0,t) = 1(t) .$$

A solution of (3) as a Volterra equation of the first kind with a variable upper limit, which was obtained in [5, 6] has the form

$$\begin{aligned} T_0(t) &= T_\mathrm{b}(t) + \frac{1}{\gamma a} \int_{-\infty}^{t} T_\mathrm{b}'(\tau) \frac{\mathrm{d}\tau}{\sqrt{\pi(t-\tau)}} = \\ &= T_\mathrm{b}(t) + \frac{1}{\gamma a} \int_{-\infty}^{t} [T_\mathrm{b}(t) - T_\mathrm{b}(\tau)] \frac{\mathrm{d}\tau}{\sqrt{\pi(t-\tau)^3}} . \end{aligned} \tag{5}$$

By substituting (5) into (1) we solve a radio thermometry problem for a homogeneous half-space [5, 6]:

$$\begin{aligned} T(z,t) &= \int_{-\infty}^{t} T_\mathrm{b}(\tau)(-z) \exp\left(-\frac{z^2}{4a^2(t-\tau)}\right) \frac{\mathrm{d}\tau}{\sqrt{4\pi a^2 (t-\tau)^3}} + \\ &+ \frac{1}{\gamma a} \int_{-\infty}^{t} T_\mathrm{b}'(\tau) \exp\left(-\frac{z^2}{4a^2(t-\tau)}\right) \frac{\mathrm{d}\tau}{\sqrt{\pi(t-\tau)}} \end{aligned} \tag{6}$$

By integrating the second term in (6) by parts, we find a formula for determination of the temperature profile by the evolution of the brightness temperature of the medium [5, 6]:

$$T(z,t) = \int\limits_{-\infty}^{t} T_{\mathrm{b}}(\tau) \exp\left(-\frac{z^2}{4a^2(t-\tau)}\right)\left[\frac{1}{\gamma}\left(\frac{z^2}{2a^2(t-\tau)}-1\right)-z\right]\frac{\mathrm{d}\tau}{\sqrt{4\pi a^2(t-\tau)^3}} . \qquad (7)$$

This formula is valid for all values of z except for $z = 0$, when integration by parts in (6) is not possible. This property, which is important for further discussion, was not mentioned in the previous papers.

Using these relations, we can obtain one more very interesting result: a formula which expresses the brightness temperature at one wavelength through the brightness temperature evolution at another wavelength. For this, we must substitute the temperature profile in the form of (6), which is expressed through the evolution of the brightness temperature T_{b1} at wavelength λ_1, into formula (3) for the brightness temperature T_{b2} at wavelength λ_2. By reversal of the integration order and calculation of the internal integral with respect to z in explicit form, we find the desired formula

$$\begin{aligned} T_{\mathrm{b}_2}(t) &= \frac{\gamma_2}{\gamma_1} T_{\mathrm{b}_1}(t) + \\ &+ \int\limits_{-\infty}^{t} T_{\mathrm{b}_1}(\tau)\left(1-\frac{\gamma_2}{\gamma_1}\right)\left\{\gamma_2 a\left[\frac{1}{\sqrt{\pi(t-\tau)}} - \gamma_2 a e^{(\gamma_2 a)^2(t-\tau)} \operatorname{erfc}\left(\gamma_2 a\sqrt{t-\tau}\right)\right]\right\} \mathrm{d}\tau , \qquad (8) \end{aligned}$$

where γ_1 and γ_2 are the absorption coefficients at wavelengths $\lambda_1$and λ_2, respectively. It is interesting to note that the use of Eq. (7) instead of Eq. (6) leads to an incorrect result, although formula (7) is invalid at only at one point $z = 0$ (because of the divergence at point $z = 0$ in the integrand).

For $\gamma_1 = \gamma_2$, we find, from (8), the obvious result $T_{b1}, = T_{b2}$. In the case $\gamma_2 << \gamma_1$, the first term in (8) vanishes, and we obtain a formula similar to relation (3), in which the role of the surface temperature is played by the brightness temperature T_{b1}.This result is physically apparent, since the thickness of the skin layer $d_2 = 1/\gamma_2$, in which thermal emission is induced at wavelength λ_2, is much greater than the thickness d_1 at wavelength λ_1, and the brightness temperature T_{b1}, indeed plays the role of surface temperature for the brightness temperature T_{b2}.

Relation (8) can be used for determination of the parameters of the medium by simultaneous measurement of thermal emission at two or more wavelengths.

3. DETERMINATION OF STATISTICAL PARAMETERS OF THERMAL RADIO EMISSION

Let the boundary condition for the temperature be a random stationary function with a given mean (To), standard deviation $<T_0>$ and autocovariance function $B_{T_0T_0}(\tau) = <(T_0(t) - <T_0>)(T_0(t+\tau) - <T_0>)>$. To obtain obvious physical results, the latter can be assigned in the form of an exponential function,

$$B_{T_0T_0}(\tau) = \sigma_{T_0}^2 \exp\left(-\left|\frac{\tau}{\tau_0}\right|\right), \qquad (9)$$

where TO is the correlation time.

The goal of further analysis is that of determining the correlation functions of the brightness temperatures of thermal emission through the parameters of the medium and statistical parameters of the temperature. It is obvious that $<T(z)> = <T_0)>$ and $<T_b> = <To>$ for average values, since all of the above integral relationships are normalized to unity.

If the temperature at the half-space boundary is a random quantity, then the above relationships, which are linear integral expressions, make it possible to develop a statistical theory of random components of the temperature distribution and thermal emission of the medium on the basis of the well-known method **in** the theory of stationary random processes for Linear systems, which leads to Wiener-Lee equations.

The results given below determine the statistical parameters of the temperature distribution and thermal emission of the medium through the statistical parameters of the surface temperature of the medium. These formulas are easily obtained from the above expressions by changing the averaging order and integrating with substitution of variables $\tau' = t - \tau$ and are represented in a form which is valid for both positive and negative half-spaces. Also, we note one more property of the covariance functions, that $B_{yx}(-\tau) = B_{xy}(\tau)$, which will be used in what follows.

Thus, it follows from (1) that the covariance function between the surface temperature T_0 and the temperature $T(z)$ at level z is given by

$$\begin{aligned} B_{T_0T}(\tau, z) &= \int_0^\infty B_{T_0T_0}(\tau - \tau')K(\tau')\mathrm{d}\tau' = \\ &= \int_0^\infty B_{T_0T_0}(\tau - \tau')\frac{|z|}{2\sqrt{\pi}\,a}\exp\left(-\frac{z^2}{4a^2\tau'}\right)\frac{\mathrm{d}\tau'}{(\tau')^{3/2}}, \end{aligned} \tag{10}$$

where $K(\tau')$ is the kernel of the integral in (1). Hence we find expressions for the variance,

$$\begin{aligned} \sigma_T^2(z) &= \int_0^\infty B_{T_0T}(\tau)K(\tau)\,\mathrm{d}\tau = \\ &= \iint_0^\infty B_{T_0T_0}(\tau - \tau')K(\tau')K(\tau)\,\mathrm{d}\tau\,\mathrm{d}\tau' = \\ &= \iint_0^\infty B_{T_0T_0}(\tau - \tau')\frac{z^2}{4\pi a^2}\exp\left[-\frac{z^2}{4a^2}\left(\frac{1}{\tau'} + \frac{1}{\tau}\right)\right]\frac{\mathrm{d}\tau\,\mathrm{d}\tau'}{(\tau\tau')^{3/2}}, \end{aligned} \tag{11}$$

for the autocovariance matrix of the temperature at level z,

$$\begin{aligned} B_{TT}(\tau, z) &= \int_0^\infty B_{T_0T}(\tau' - \tau)K(\tau')\mathrm{d}\tau' = \\ &= \iint_0^\infty B_{T_0T_0}(\tau' - \tau'' - \tau)K(\tau')K(\tau'')\,\mathrm{d}\tau'\,\mathrm{d}\tau'' = \\ &= \iint_0^\infty B_{T_0T_0}(\tau' - \tau'' - \tau)\frac{z^2}{4\pi a^2}\exp\left[-\frac{z^2}{4a^2}\left(\frac{1}{\tau'} + \frac{1}{\tau''}\right)\right]\frac{\mathrm{d}\tau'\,\mathrm{d}\tau''}{(\tau'\tau'')^{3/2}} \end{aligned} \tag{12}$$

and for the interlevel covariance matrix between the temperature T_1 at level z_1 and the temperature T_2 at level z_2 (T_2 plays the role of surface temperature in the kernel of integral (1) in this case),

$$\begin{aligned} B_{T_2T_1}(z_2, z_1, \tau) &= \int_0^\infty B_{T_2T_2}(\tau - \tau')K(\tau')\,\mathrm{d}\tau' = \iiint_0^\infty \frac{\mathrm{d}\tau'\,\mathrm{d}\tau''\,\mathrm{d}\tau'''}{(\tau'\tau''\tau''')^{3/2}} \times \\ &\times B_{T_0T_0}(\tau' - \tau'' + \tau''' - \tau)\frac{z_2^2|z_1 - z_2|}{8\pi^{3/2}a^3}\exp\left[-\frac{z_2^2}{4a^2}\left(\frac{1}{\tau'} + \frac{1}{\tau''}\right) - \frac{(z_1 - z_2)^2}{4a^2}\frac{1}{\tau'''}\right] \end{aligned} \tag{13}$$

The covariance function between the surface temperature and brightness temperature of the thermal radio emission of the medium is described by the expression

$$B_{T_0 T_b}(\tau) = \int_0^{\infty} B_{T_0 T_0}(\tau - \tau') K_1(\tau')\, d\tau' =$$

$$= \int_0^{\infty} B_{T_0 T_0}(\tau - \tau') \frac{\gamma a}{\sqrt{\pi \tau'}} \left[1 - \sqrt{\pi}(\gamma a)\sqrt{\tau'} e^{(\gamma a)^2 \tau'} \operatorname{erfc}(\gamma a \sqrt{\tau'})\right] d\tau', \qquad (14)$$

where K_1 is the kernel of the integral in (3) or in (4). Prom (14), we find expressions for the variance

$$\sigma_{T_b}^2 = \int_0^{\infty} B_{T_0 T_b}(\tau) K_1(\tau)\, d\tau = \iint_0^{\infty} B_{T_0 T_b}(\tau - \tau') K_1(\tau') K_1(\tau)\, d\tau\, d\tau' =$$

$$= \iint_0^{\infty} d\tau\, d\tau'\, B_{T_0 T_0}(\tau - \tau') \frac{(\gamma a)^2}{\pi \sqrt{\tau \tau'}} \times$$

$$\times \left[1 - \sqrt{\pi}(\gamma a)\sqrt{\tau'} e^{(\gamma a)^2 \tau'} \operatorname{erfc}(\gamma a \sqrt{\tau'})\right] \left[1 - \sqrt{\pi}(\gamma a)\sqrt{\tau} e^{(\gamma a)^2 \tau} \operatorname{erfc}(\gamma a \sqrt{\tau})\right], \qquad (15)$$

and for the autocovariance function of the brightness temperature

$$B_{T_b T_b}(\tau) = \iint_0^{\infty} B_{T_b T_0}(\tau - \tau') K_1(\tau')\, d\tau' =$$

$$= \iint_0^{\infty} B_{T_0 T_0}(\tau' - \tau'' - \tau) K(\tau') K(\tau'')\, d\tau'\, d\tau'' =$$

$$= \iint_0^{\infty} d\tau'\, d\tau''\, B_{T_0 T_0}(\tau' - \tau'' - \tau) \frac{(\gamma a)^2}{\pi \sqrt{\tau' \tau''}} \times$$

$$\times \left[1 - \sqrt{\pi}(\gamma a)\sqrt{\tau'} e^{(\gamma a)^2 \tau'} \operatorname{erfc}(\gamma a \sqrt{\tau'})\right] \left[1 - \sqrt{\pi}(\gamma a)\sqrt{\tau''} e^{(\gamma a)^2 \tau''} \operatorname{erfc}(\gamma a \sqrt{\tau''})\right]. \qquad (16)$$

Formula (16) can be used to find an expression for the covariance matrix between brightness temperatures T_{b1} and T_{b2} at two different wavelengths, λ_1 and λ_2,

$$B_{T_{b_1} T_{b_2}}(\lambda_1, \lambda_2, \tau) = \iint_0^{\infty} B_{T_{b_1} T_{b_1}}(\tau - \tau') K_1(\tau')\, d\tau' = \frac{\gamma_2}{\gamma_1} B_{T_{b_1} T_{b_1}}(\tau) +$$

$$+ \left(1 - \frac{\gamma_2}{\gamma_1}\right) \int_0^{\infty} B_{T_{b_1} T_{b_1}}(\tau - \tau') \frac{\gamma_2 a}{\sqrt{\pi \tau'}} \left[1 - \sqrt{\pi}(\gamma_2 a)\sqrt{\tau'} e^{(\gamma_2 a)^2 \tau'} \operatorname{erfc}(\gamma_2 a \sqrt{\tau'})\right] d\tau', \qquad (17)$$

from (8) and to obtain a formula for the covariance matrix between the brightness temperature and the temperature at level z, from (6)

$$B_{T_b T}(\tau, z) = \int_0^{\infty} B_{T_b T_b}(\tau - \tau') \frac{|z|}{2\sqrt{\pi} a} \exp\left(-\frac{z^2}{4a^2 \tau'}\right) \frac{d\tau'}{(\tau')^{3/2}} +$$

$$+ \frac{1}{\gamma a} \int_0^{\infty} \frac{\partial B_{T_b T_b}}{\partial \tau'}(\tau - \tau') \exp\left(-\frac{z^2}{4a^2 \tau'}\right) \frac{d\tau'}{\sqrt{\pi}(\tau')^{1/2}}. \qquad (18)$$

4. REGRESSION FORMULAS. ESTIMATES FOR SOIL AND ATMOSPHERE RADIO THERMOMETRY

The formulas which we obtained are of interest from the point of view of the widely used statistical methods for determination of temperature profiles by thermal radio emission or surface temperature [12-14], since earlier the parameters considered in the previous section of this paper were defined only empirically on the basis of a statistical analysis of measurements. The results presented here allow the necessary quantities to be calculated by exact formulas. Moreover, the physical meaning of those quantities has been clarified.

Relation (10) is easily used for regression estimation of the temperature profile by the surface temperature:

$$T(z,t) = \langle T_0 \rangle + \frac{B_{T_0 T}(\tau, z)}{\sigma^2_{T_0}} [T_0(t-\tau) - \langle T_0 \rangle] \tag{19}$$

In a similar way, the temperature profile can be estimated by the brightness temperature of the medium from (14):

$$T(z,t) = \langle T_0 \rangle + \frac{B_{T_b T}(\tau, z)}{\sigma^2_{T_b}} [T_b(t-\tau) - \langle T_0 \rangle] \, , \tag{20}$$

or by using more complicated methods of multidimensional regression and statistical regularization [12-14]. Of course, the other covariance functions presented above can also be used in the same fashion to evaluate the corresponding quantities. It is well known that the error $\sigma_{y/x}$ in the regression estimate of y by x is determined by the correlation factor $R_{xy} = B_{xy}/\sigma_x\sigma_y$ between these quantities and their variances. In other words, this error can be calculated on the basis of the reduced expressions

$$\sigma^2_{y/x} = \sigma^2_y (1 - R^2_{xy}) \, . \tag{21}$$

Prom the expressions for the correlation functions it is seen that the latter are not symmetrical with respect to $\tau = 0$; moreover, they do not reach a maximum at this point, i.e., the prediction for the future is not symmetrical to the prediction for the past with respect to time shift, and the prediction of the profile by current values of the earth-surface or brightness temperature, which is known as "optimal extrapolation" in the literature, is not optimal in fact. It is seen from (21) that an optimal estimate of a predicted quantity at time t is an estimate of that quantity by the corresponding predictor at time $(t - \tau_m)$ at which the function $R_{xy}(\tau)$ and, therefore, $B_{xy}(\tau)$ reach maxima for τ_m. The condition from which we determine the value τ_m is, of course, $dB(\tau)/d\tau = 0$, and the corresponding equations are easily obtained from these expressions for the covariance functions. Specifically, in temperature-profile prediction each value of z will correspond to a certain value of $\tau_m(z)$, and an optimal extrapolation is reached in prediction by the previous value of surface temperature rather than by its current value. Obviously, the time shift increases with z, i.e., $\tau_m(z)$ is a monotonically increasing function. This property of the regression estimate is physically apparent: The surface perturbation of temperature acts on the temperature of the deeper layers not instantaneously but through a thermal conductivity mechanism with a delay which increases with depth.

It can be assumed that in empirical covariance functions describing the temperature of the actual atmosphere, in which physical conditions do not correspond perfectly to the model in question, the maximum correlation will nevertheless also be reached at earth surface temperatures in the past, especially if we consider the boundary layer. It is also obvious that from these equations we can find statistical estimates not only for the future ($\tau > 0$) and current value ($\tau = O$) of the quantity of interest but also for its value in the past ($\tau < 0$).

Simple analytical results can be obtained for an exponential covariance function of form (9). Specifically, it follows from (10) that

$$B_{T_0 T}(\tau) = \int_0^\infty \frac{\sigma^2_{T_0}|z|}{2\sqrt{\pi}a} \exp\left(-\frac{z^2}{4a^2\tau} - \frac{|\tau - \tau'|}{\tau_0}\right) \frac{d\tau'}{(\tau')^{3/2}} = \sigma^2_{T_0} \exp\left(-\frac{|z|}{a\sqrt{\tau_0}} - \left|\frac{\tau}{\tau_0}\right|\right) \, , \tag{22}$$

for τ ≤ 0 and

$$B_{T_0T}(\tau)=\frac{\sigma_{T_0}^2|z|}{2\sqrt{\pi a}}\left[\int_0^{\tau}\exp\left(-\frac{z^2}{4a^2\tau}+\frac{\tau'}{\tau_0}\right)\frac{d\tau'}{(\tau')^{3/2}}\cdot\exp\left(-\frac{\tau}{\tau_0}\right)+\right.$$
$$\left.+\int_{\tau}^{\infty}\exp\left(-\frac{z^2}{4a^2\tau}-\frac{\tau'}{\tau_0}\right)\frac{d\tau'}{(\tau')^{3/2}}\cdot\exp\left(\frac{\tau}{\tau_0}\right)\right]. \tag{22a}$$

for τ > 0.

In particular,

$$B_{T_0T}(0)=\sigma_{T_0}^2\exp\left(-\frac{|z|}{a\sqrt{\tau_0}}\right). \tag{23}$$

It is seen that there is a characteristic correlation distance of the temperature profile with surface value $\Lambda = a\sqrt{\tau_0}$ which can serve as a definition of the atmospheric boundary layer. In the case at hand, the equation for determination of the time shift of optimal extrapolation $\tau_m(z)$ is given by

$$\int_{\tau}^{\infty}\exp\left(-\frac{z^2}{4a^2\tau}-\frac{\tau'}{\tau_0}\right)\frac{d\tau'}{(\tau')^{3/2}}\cdot\exp\left(\frac{\tau}{\tau_0}\right)-\int_0^{\tau}\exp\left(-\frac{z^2}{4a^2\tau}+\frac{\tau'}{\tau_0}\right)\frac{d\tau'}{(\tau')^{3/2}}\cdot\exp\left(-\frac{\tau}{\tau_0}\right)=0. \tag{24}$$

It follows from (14) that

$$B_{T_0T_b}(\tau)=\int_0^{\infty}\frac{\sigma_{T_0}^2\gamma a}{\sqrt{\pi\tau'}}\left[1-\sqrt{\pi}(\gamma a)\sqrt{\tau'}\exp[(\gamma a)^2\tau']\operatorname{erfc}(\gamma a\sqrt{\tau'})\right]\exp\left(-\frac{|\tau-\tau'|}{\tau_0}\right)d\tau'. \tag{25}$$

Calculating the integral in (25), we find

$$B_{T_0T_b}(\tau)=\sigma_{T_0}^2\frac{\sqrt{\tau_0/\Gamma}}{1+\sqrt{\tau_0/\Gamma}}\exp\left(-\left|\frac{\tau}{\tau_0}\right|\right), \tag{26}$$

for τ ≤ 0 and

$$B_{T_0T_b}(\tau)=\sigma_{T_0}^2\left\{\exp\left(-\frac{\tau}{\tau_0}\right)+\right.$$
$$+\frac{1}{\tau_0}\exp\left(-\frac{\tau}{\tau_0}\right)\frac{1}{(\gamma a)^2+(1/\tau_0)}\left[\gamma a\sqrt{\tau}\,{}_1F_1\left(\frac{1}{2},\frac{3}{2},\frac{\tau}{\tau_0}\right)+\exp\left[-\left((\gamma a)^2+\frac{1}{\tau_0}\right)\tau\right]\operatorname{erfc}(\gamma a\sqrt{\tau})-1\right]-$$
$$\left.-\frac{1}{\tau_0}\exp\left(\frac{\tau}{\tau_0}\right)\frac{1}{(\gamma a)^2-(1/\tau_0)}\left[\gamma a\sqrt{\tau_0}\operatorname{erfc}\left(\frac{\tau}{\tau_0}\right)-\exp\left[\left((\gamma a)^2-\frac{1}{\tau_0}\right)\tau\right]\operatorname{erfc}(\gamma a\sqrt{\tau})\right]\right\}, \tag{26a}$$

for τ > 0. Here ${}_1F_1$ is a degenerate hypergeometric function and $\Gamma = 1/(\gamma a)^2$ is the characteristic time scale, which determines the heating of the medium to depth $d = 1/\gamma$ of the skin layer. It follows from (26) that

$$B_{T_0T_b}(0)=\sigma_{T_0}^2\frac{\sqrt{\tau_0/\Gamma}}{1+\sqrt{\tau_0/\Gamma}}. \tag{27}$$

It is seen that the brightness temperature-surface temperature correlation is determined by the proportional time of surface temperature correlation and heating to the skin-layer depth at the corresponding wavelength. We have $B_{T0Tb}(0) = \sigma_{T0}^2$ if $\tau_0/\Gamma >> 1$ and $B_{T0Tb}(0) = 0$ if $\tau_0/\Gamma << 1$. This result is perfectly clear. If the medium is heated and cooled to the skin-layer depth during the period of surface temperature correlation, then its brightness temperature is completely correlated with its surface temperature; otherwise, variations of these quantities are not correlated.

Considering the problem of the possible use of this theory for study of the atmosphere and soil, we note the following. In the case of soil (homogeneous soil in particular) it should be expected that the random components of the temperature distribution and thermal radio emission, which are superimposed on periodic diurnal and seasonal variations, can be described correctly enough by this theory. The situation is different for the atmosphere, since the random component of the temperature profile is affected not only by thermal conductivity from the earth's surface but also by processes of advection, liberation of latent heat, and transfer and absorption of IR radiation. Moreover, the conditions of homogeneity of the medium and time independence of its parameters (turbulent thermal conductivity in particular) are not satisfied in the atmosphere, as a rule. However, this statistical theory is useful to describe radiation in strong lines of absorption in the boundary layer of the atmosphere, for example in oxygen lines at a frequency of 60 GHz, where the atmosphere is homogeneous with respect to the absorption factor. This was shown by the successful use of the initial formulas in [5, 6]. Also, it can be hoped (although this will require verification) that the vertical scale of temperature correlation Л, which is determined by expression (23), is also meaningful if the atmosphere is inhomogeneous or has time-dependent parameters, provided that the vertical transfer of heat is determined by turbulent diffusion and the mean turbulent thermal conductivity coefficient is used for the boundary layer. Conclusions on the conditions and applicability limits of this theory can be drawn by comparing the results of theoretical calculations and the empirical covariance functions.

We now give some estimates for the above media based on the equations which we obtained.

Usually, the correlation time of surface temperature is about three days, i.e., $\tau_0 \approx 2.6 \cdot 10^s$ sec (although, strictly speaking, temperature variation is a non-stationary process and its structure function also increases beyond this time period). Radiometric studies of the temperature dynamics of soil and of the atmospheric boundary layer are presented in [5, 6]. For soil, the parameters of the media are $a^2 = 1.0 \cdot 10^{-3}\text{cm}^2/\text{sec}$ and $d = 1/\gamma \approx \lambda$ (λ is the wavelength), the time parameter $\Gamma = 1/(\gamma a)^2$ varies from 10 min at $\lambda = 0.8$ cm to 50 h at $\lambda = 13$ cm, and the characteristic correlation depth in (20) $\Lambda \approx 16$ cm. For the atmosphere, $a^2 = 7.0 \cdot 10^3\text{cm}^2/\text{sec}$, $d = 3.0 \cdot 10^4 \sin(\theta)$ cm (θ is the angle at which atmospheric radiation is received), the parameter Γ varies from 16 min at a measurement angle of 5° to 35 h in the zenith direction, and the correlation height $\Lambda \approx$ 430 m.

Under natural conditions for different types of soil, these parameters lie within the limits of $a^2 = 10^{-3}\text{-}10^{-2}\text{cm}^2/\text{sec}$, and $d = 0.1\text{-}15\lambda$, correspondingly, the time Γ lies in the range from several fractions of a second for a water surface at millimeter wavelengths to several years for ice at decimeter wavelengths, and the correlation depth can be $\Lambda = 15\text{-}60$ cm. In the atmosphere, $a^2 = 10^3 - 10^6\text{cm}^2/\text{sec}$, $d = 3.0 \cdot 10^4\sin(\theta)$ cm (at a frequency of 60 GHz), the characteristic time of skin-layer heating Γ can vary from 1 min at an angle of 5° to 10 days in the zenith direction, and the correlation scale between brightness temperature and earth-surface temperature, i.e., the depth of the boundary layer, can assume values $\Lambda = 100$ m - 3 km.

5. CONCLUSION

In this paper, we developed further the theory of simultaneous solution of the thermal emission transfer and thermal conductivity equations. The relationship between the brightness temperatures of the medium at two different wavelengths is found. A statistical theory of temperature distribution and thermal radio emission of the medium (half-space) is devised on the basis of a stochastic approach. The equations for joint and autocorrelation functions of the temperature profile and brightness temperatures of thermal radio emission are derived. Through a numerical procedure, this theory can be used in full measure for both theoretical estimates and for determination of some parameters of the medium and its thermal emission by experimental data. The standard variations of the temperature profile and brightness temperatures can be calculated from the statistical parameters of the surface temperature. Interesting results can be obtained by comparing the statistical estimates of the temperature profile with data reconstructed on the basis of Tikhonov's universal method [10-11] to ascertain the applicability range of this theory under various conditions.

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